4-8

a) $X_1(f) = \frac{1}{2}\left[e^{+j2\pi f} + e^{j\pi f} + e^{-j\pi f} + e^{-j2\pi f}\right]$

$= \cos 2\pi f + \cos \pi f$

b) $x_2(t) = \text{sinc}(t)\, u(t)$

$$X_2(f) = \Pi(f) * \left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right]$$

$$= \Pi(f) * \frac{1}{j2\pi f} + \frac{1}{2}\Pi(f)$$

$\left\{ \begin{array}{l} \text{note:} \\ x(t) * \delta(t) = \delta(t) * x(t) \\ \quad = x(t) \end{array} \right\}$

$$= \int_{-\infty}^{\infty} \frac{1}{j2\pi\lambda}\, \pi(f-\lambda)\, d\lambda + \frac{1}{2}\pi(f)$$

$$= \int_{f-\frac{1}{2}}^{f+\frac{1}{2}} \frac{1}{j2\pi\lambda}\, d\lambda + \frac{1}{2}\pi(f)$$

$$= \frac{1}{j2\pi} \ln\left|\frac{f+\frac{1}{2}}{f-\frac{1}{2}}\right| + \frac{1}{2}\,\Pi(f)$$

c) $x_3(t) = \mathrm{sinc}(t)\,\mathrm{sgn}(t)$

$$X_3(f) = \Pi(f) * \frac{1}{j\pi f}$$

$$= \frac{1}{j\pi} \ln\left|\frac{f+\frac{1}{2}}{f-\frac{1}{2}}\right| \quad , \left(\begin{array}{c}\text{using the}\\ \text{previous result}\end{array}\right).$$

d) $x_4(t) = e^{-|t|}\,u(t) = e^{-t}\,u(t)$

$$X_4(f) = \frac{1}{1+j2\pi f}$$

e) $x_5(t) = e^{-|t|}\,\mathrm{sgn}(t) = -e^{t}\,u(-t) + e^{-t}\,u(t)$

$$= -\frac{1}{1-j2\pi f} + \frac{1}{1+j2\pi f} = \frac{-j4\pi f}{1+4\pi^2 f^2}$$

General notes regarding symmetry of $x(t)$

part a) $x_1(t)$, real and even $\Rightarrow X_1(f)$, real and even.

Part b) $x_2(t)$, real (not even or odd) $\Rightarrow$ $X_2(f)$, complex.

Part c) $x_3(t)$, real and odd $\Rightarrow X_3(f)$, imaginary and odd.

Part d) $x_4(t)$, real $\Rightarrow X_4(f)$ complex.

Part e) $x_5(t)$, real and odd $\Rightarrow X_5(f)$, imaginary, odd.

4-9

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} x(t)\cos 2\pi ft\, dt - j\int_{-\infty}^{\infty} x(t)\sin 2\pi ft\, dt$$

$$X(-f) = \int_{-\infty}^{\infty} x(t)\cos 2\pi ft\, dt + j\int_{-\infty}^{\infty} x(t)\sin 2\pi ft\, dt$$

[Provided $x(t)$ is real $\Rightarrow X(-f) = X^*(f)$]

$$\because \quad X(f) = |X(f)|\, e^{j\theta(f)}$$

$$\because \quad X(-f) = |X(-f)|\, e^{j\theta(-f)}$$

$$\therefore \quad |X(-f)|\, e^{j\theta(-f)} = |X(f)|\, e^{-j\theta(f)}$$

$$\therefore \quad |X(f)| = |X(-f)|$$

$$\therefore \quad \theta(f) = -\theta(-f)$$

4-20

a) Transform should be real and even.

$$x_a(t) = r(t+2) - 2r(t+1) + 2r(t) - 2r(t-1) + r(t-2)$$

$$x_a''(t) = \delta(t+2) - 2\delta(t+1) + 2\delta(t) - 2\delta(t-1) + \delta(t-2)$$

$$X_a(f) = \frac{1}{(j2\pi f)^2}\left[e^{j4\pi f} - 2e^{j2\pi f} + 2 - 2e^{-j2\pi f} + e^{-j4\pi f}\right]$$

$$= \frac{-1}{4\pi^2 f^2}\left[2\cos 4\pi f - 4\cos 2\pi f + 2\right]$$

$$= \frac{-1}{4\pi^2 f^2}\left[4\cos^2 2\pi f - 4\cos 2\pi f\right]$$

$$= \frac{\cos 2\pi f}{\pi^2 f^2}\left[1 - \cos 2\pi f\right]$$

$$= \frac{2(\cos 2\pi f)(\sin \pi f)^2}{\pi^2 f^2} = 2\cos 2\pi f \,\operatorname{sinc}^2(f)$$

b) Transform should be imaginary and odd.

$$x_b(t) = -r(t+2) + 2r(t+1) - 2r(t-1) + r(t-2)$$

$$x_b''(t) = -\delta(t+2) + 2\delta(t+1) - 2\delta(t-1) + \delta(t-2)$$

$$X_b(f) = \frac{1}{(j2\pi f)^2}\left[-e^{j4\pi f} + 2e^{j2\pi f} - 2e^{-j2\pi f} + e^{-j4\pi f}\right]$$

$$= \frac{4j\sin 2\pi f - 2j\sin 4\pi f}{-4\pi^2 f^2}$$

$$= 2j\,\frac{\sin 4\pi f - 2\sin 2\pi f}{4\pi^2 f^2}$$

$$= j\,\frac{2\sin 2\pi f\cos 2\pi f - 2\sin 2\pi f}{2\pi^2 f^2}$$

$$= j\,\frac{\sin 2\pi f}{\pi^2 f^2}\left[\cos 2\pi f - 1\right]$$

$$= -2j\,\frac{\sin 2\pi f}{\pi^2 f^2}\,\sin^2\pi f$$

$$= -2j\,\sin 2\pi f\,\operatorname{sinc}^2(f)$$

25 a)

$$y_1(t) = x_1(t) * h_1(t)$$

$$Y_1(f) = X_1(f)\, H_1(f)$$

$$= \frac{1}{(\alpha + j2\pi f)^2} \cdot \frac{1}{(\beta + j2\pi f)}$$

$$= \frac{\left(\frac{1}{\beta - \alpha}\right)}{(\alpha + j2\pi f)^2} - \frac{\frac{1}{(\alpha-\beta)^2}}{\alpha + j2\pi f} + \frac{\frac{1}{(\alpha-\beta)^2}}{\beta + j2\pi f}$$

[The above expansion is called partial fraction expansion, which will be studied in a future lecture].

$$y_1(t) = \mathcal{F}^{-1}\left[Y_1(f)\right]$$

$$= \left(\frac{1}{\beta - \alpha}\right) t\, e^{-\alpha t} u(t) - \frac{1}{(\alpha - \beta)^2} e^{-\alpha t} u(t)$$

$$+ \frac{1}{(\alpha - \beta)^2} e^{-\beta t} u(t)$$