8.3 Z- Transform

Similar to the Laplace Transform, we will develop a transform for the analysis and Synthesis (not in this course) of discrete system

Let

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(t)\,\delta(t-nT) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

Assuming: $x(t) \equiv 0 \qquad t<0$

$$\Rightarrow \quad x_s(t) = \sum_{n=0}^{\infty} x(nT)\,\delta(t-nT)$$

taking Laplace Transform,

$$X_s(s) = \int_0^{\infty} \sum_{n=0}^{\infty} x(nT)\,\delta(t-nT)\,e^{-st}\,dt$$

$$\Rightarrow \quad X_s(t) = \sum_{n=0}^{\infty} x(nT)\,\delta(t-nT)$$

taking Laplace Transform,

$$X_s(s) = \int_0^{\infty} \sum_{n=0}^{\infty} x(nT)\,\delta(t-nT)\,e^{-st}\,dt$$

interchanging sumation and integration,

$$X_s(s) = \sum_{n=0}^{\infty} x(nT) \int_0^{\infty} \delta(t-nT)\,e^{-st}\,dt = \sum_{n=0}^{\infty} x(nT)\,e^{-snT}$$

define $z = e^{sT} \Rightarrow X_s(z) = \sum_{n=0}^{\infty} x(nT)\,z^{-n}$

Z-Transform of the Sequence samples x(nT)

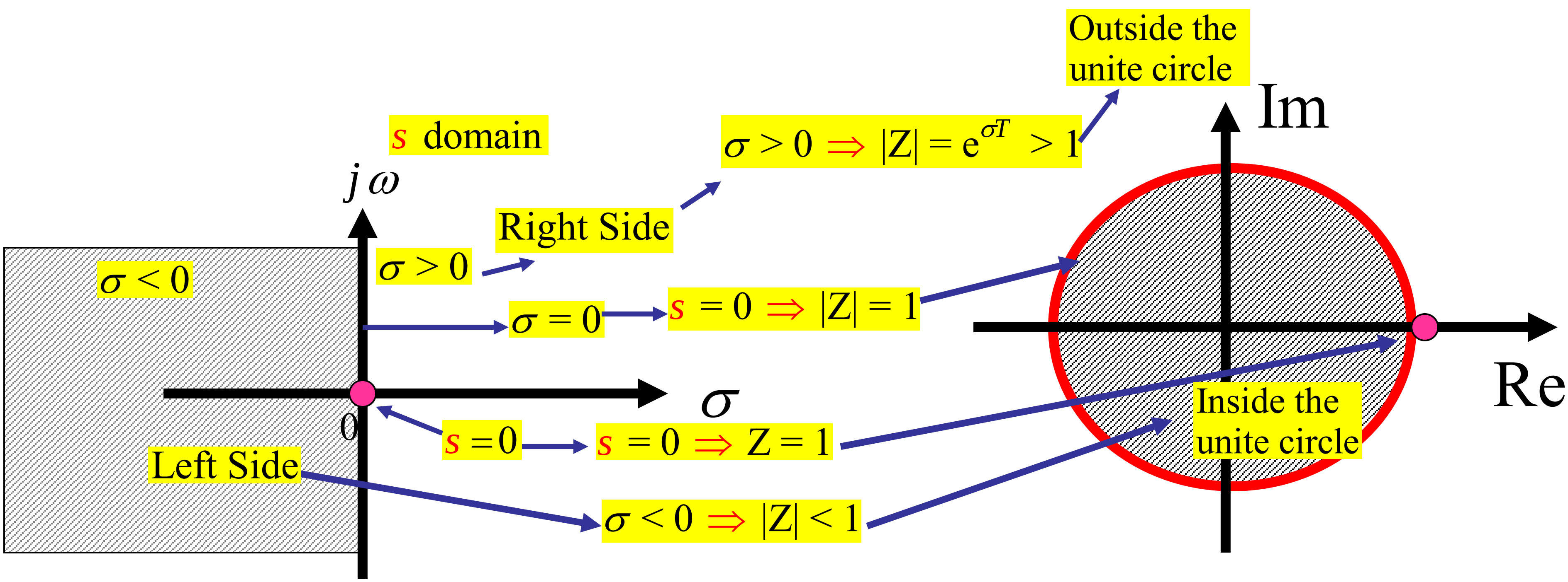
Z-Transform of the Sequence samples x(nT)

$$X_s(z) = \sum_{n=0}^{\infty} x(nT)\, z^{-n}$$

The subscript "s" in $X_s(z)$ can be deleted

$$X(z) = \sum_{n=0}^{\infty} x(nT)\, z^{-n}$$

Since $s = \sigma + j\omega \Rightarrow z = e^{sT} = e^{\sigma T} e^{j\omega T} \Rightarrow |z| = e^{\sigma T}$



In the Z-transform,

$$X(z)=\sum_{n=0}^{\infty} x(nT)z^{-n}=x(0)z^{0}+x(1)z^{-1}+x(2)z^{-2}+\cdots$$

The coefficient x(nT) denotes the sample values and z^{-n} denote the Sample occurs n sample periods after t = 0

Example $10z^{-15}$ Mean a sample having a value 10 Occurring 15 samples period after t = 0

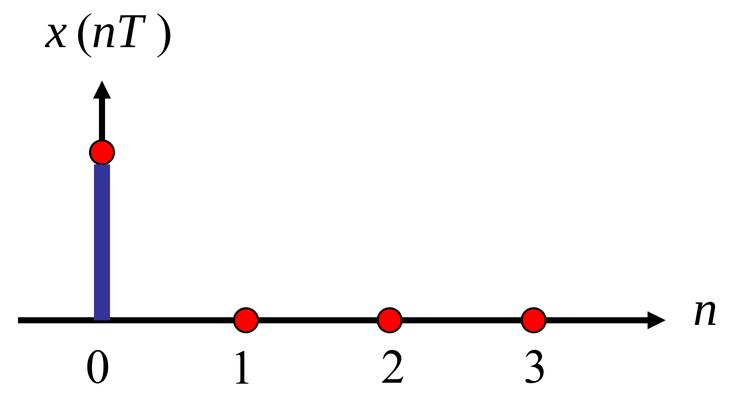
Note $z=e^{sT}$ Which imply T-second shift

$z^{-n}=e^{-snT}$ Which imply nT-second shift

Example 8-4

Define the unite impulse sequence by ,

$$x(nT)=\begin{cases}1 & n=0\\ 0 & n\neq 0\end{cases}\triangleq\delta(nT)$$



Note : the unit impulse here (the discrete) is different from the impulse $\delta(t)$

$$\delta(n)\Big|_{n=0}=1 \qquad \delta(t)\Big|_{t=0}=\infty$$

$$X(z)=\sum_{n=0}^{\infty}x(nT)z^{-n}=(1)z^{0}+(0)z^{-1}+(0)z^{-2}+\cdots=1$$

$$\delta(t)\underset{\text{Laplace}}{\Leftrightarrow}1 \qquad \delta(n)\underset{\text{Z}}{\Leftrightarrow}1$$

Since $$\delta(n-k)=\begin{cases}1 & n=k\\ 0 & n\neq k\end{cases}$$

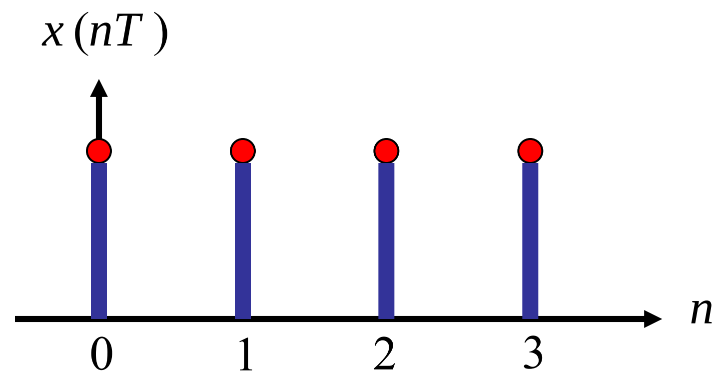
Thus $$x(nT)=\sum_{k=-\infty}^{\infty} x(kT)\delta(n-k)$$

$$g(t_0)=\int_{-\infty}^{\infty} g(t)\delta(t-t_0)dt$$ Sifting property

Example 8-5

Define the unite step by the sample values

$$x(nT) = 1 \qquad n \geq 0$$



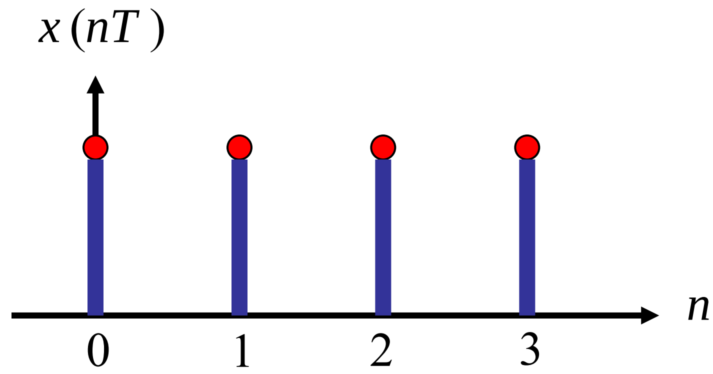
$$X(z) = \sum_{n=0}^{\infty} x(nT) z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

since $\sum_{n=0}^{\infty} x^{n} = \frac{1}{1-x}$ for |x| < 1

$$\Rightarrow X(z) = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^{n} = \frac{1}{1-(z^{-1})} = \frac{z}{z-1}$$

$$|z^{-1}| < 1 \Rightarrow |z| > 1$$

$$x(nT) = 1 \qquad n \geq 0$$



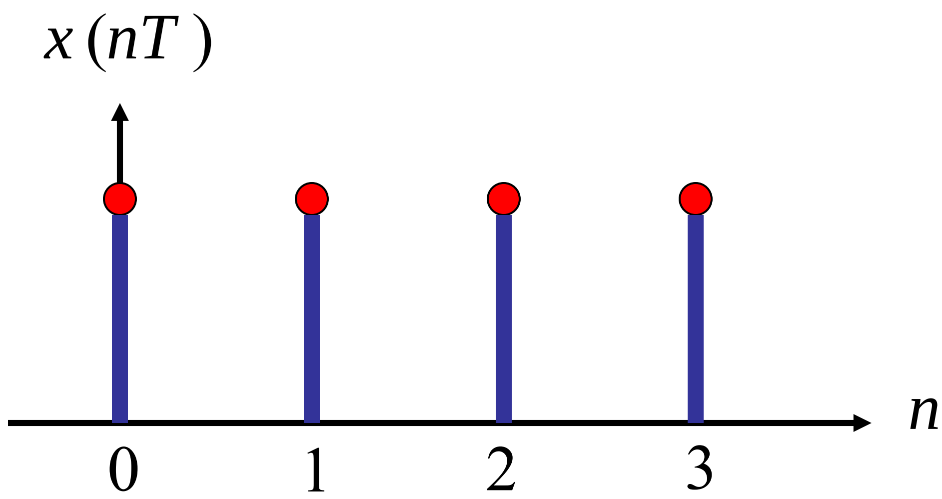
$$X(z) = \frac{z}{z-1} \qquad |z^{-1}| < 1 \Rightarrow |z| > 1$$

Note $X(z)$ has a pole at $z = 1$

$z = 1$ correspond to $s = 0$

step function $u(t) \Leftrightarrow \frac{1}{s}$ has a pole at $s = 0$

$$x(nT) = 1 \qquad n \geq 0$$



The discrete unit step defined above is not simply a sampled u(t) unless u(0) = 1

u(t) is not continuous at t = 0

The discrete unit step is denoted u(n)

Z- Transform Properties

<u>(1) Linearity</u> Z- Transform is Linear operator

If $x_1(nT) \Leftrightarrow X_1(z) \qquad x_2(nT) \Leftrightarrow X_2(z)$

<u>Then</u> $\mathrm{A}x_1(nT) + \mathrm{B}x_2(nT) \Leftrightarrow \mathrm{A}X_1(z) + \mathrm{B}X_2(z)$

Proof

$$Z[\mathrm{A}x_1(nT) + \mathrm{B}x_2(nT)] = \sum_{n=0}^{\infty}[\mathrm{A}x_1(nT) + \mathrm{B}x_2(nT)]z^{-n}$$

$$= \sum_{n=0}^{\infty}\mathrm{A}x_1(nT)\, z^{-n} + \sum_{n=0}^{\infty}\mathrm{B}x_2(nT) z^{-n}$$

$$= \mathrm{A}\sum_{n=0}^{\infty}x_1(nT)\, z^{-n} + \mathrm{B}\sum_{n=0}^{\infty}x_2(nT) z^{-n}$$

$$= AX_1(z) + BX_2(z)$$

(2) Initial value and Final value theorem

Initial value theorem $x(0) = \lim_{z \to \infty} X(z)$

Proof $X(z) = \sum_{n=0}^{\infty} x(nT) z^{-n} = x(0) + \sum_{n=1}^{\infty} x(nT) z^{-n}$

$$\lim_{z \to \infty} X(z) = \lim_{z \to \infty} x(0) + \lim_{z \to \infty} \sum_{n=1}^{\infty} x(nT) z^{-n}$$

$$= x(0) + \sum_{n=1}^{\infty} x(nT) \lim_{z \to \infty} \frac{1}{z^{n}}$$

$$= x(0) + \sum_{n=1}^{\infty} x(nT)(0) = x(0)$$

Final value theorem

$$x(\infty) = \lim_{z \to 1} (1 - z^{-1}) X(z) = \lim_{z \to 1} \frac{z-1}{z} X(z)$$

Proof (Difficult to show)

Inverse Z-Transform

Since

$$X(z)=\sum_{n=0}^{\infty}x(nT)z^{-n}=x(0)z^{0}+x(1)z^{-1}+x(2)z^{-2}+\cdots$$

Therefore , if we can put $X(z)$ into the form shown above,

Then we can determine $x(nT)$ by inspection

$x(nT)$ will be the coefficients of the polynomial of $X(z)$

Example 8-8

$$X(z) = \frac{z^2}{(z-1)(z-0.2)} = \frac{z^2}{z^2 - 1.2z + 0.2} = \frac{1}{1 - 1.2z^{-1} + 0.2z^{-2}}$$

Using polynomial division, we get

$$X(z) = 1 + 1.2z^{-1} + 1.24z^{-2} + 1.248z^{-3} + \cdots$$

Therefore

$$x(0) = 1 \quad x(1) = 1.2 \quad x(2) = 1.24 \quad x(3) = 1.248$$

The disadvantage of this method is that ,
we do not get $x(nT)$ in **closed form**

$$X(z)=\frac{z^2}{(z-1)(z-0.2)}=\frac{z^2}{z^2-1.2z+0.2}=\frac{1}{1-1.2z^{-1}+0.2z^{-2}}$$

$$X(z)=1+1.2z^{-1}+1.24z^{-2}+1.248z^{-3}+\cdots$$

$$x(0)=1 \quad x(1)=1.2 \quad x(2)=1.24 \quad x(3)=1.248$$

From Initial value theorem

$$\mathrm{x}(0)=\lim_{z\to\infty} X(z)=\lim_{z\to\infty}(1)+\lim_{z\to\infty}(\underbrace{1.2z^{-1}}_{0})+\lim_{z\to\infty}(\underbrace{1.24z^{-2}}_{0})+\lim_{z\to\infty}(\underbrace{1.248z^{-3}}_{0})+\cdots\boxed{=1}$$

Final value theorem $\mathrm{x}(\infty)=\lim_{z\to 1}(1-z^{-1})X(z)=\lim_{z\to 1}\frac{z-1}{z}X(z)$

$$\mathrm{x}(\infty)=\lim_{z\to 1}\frac{\cancel{z-1}}{\cancel{z}}\,\frac{\cancel{z^2}}{(\cancel{z-1})(z-0.2)}=\lim_{z\to 1}\frac{z}{(z-0.2)}\boxed{=1.25}$$

Using the Z-Transform Table

Example 8-9

Let $X(z) = \dfrac{1}{1-1.2z^{-1}+0.2z^{-2}}$ Find $x(n)$?

$$X(z) = \frac{z^2}{z^2-1.2z+0.2} = \frac{z^2}{(z-1)(z-0.2)}$$

Since the degree of the **Numerator** equal the degree of the **denominator**

Polynomial division $X(z) = 1 + \dfrac{1.2z-0.2}{(z-1)(z-0.2)}$

$$= 1 + \boxed{\frac{1.25}{z-1}} - \boxed{\frac{0.05}{z-0.2}}$$

This form is not available on the table

Now $$\frac{X(z)}{z}=\frac{z}{(z-1)(z-0.2)}$$

Now the degree of the **Numerator** less than the degree of the **denominator**

Using partial fraction , we have

$$\frac{X(z)}{z}=\frac{1.25}{z-1}-\frac{0.05}{z-0.2} \Rightarrow X(z)=\frac{1.25z}{z-1}-\frac{0.25z}{z-0.2}$$

$$X(z)=\frac{1.25}{1-z^{-1}}-\frac{0.25}{1-0.2z^{-1}}$$

$$\Rightarrow x(nT)=1.25-0.25(0.2)^{n} \qquad n\geq 0$$

Table 8-1

$$x(0)=1$$
$$x(1)=1.2$$
$$x(2)=1.24$$
$$x(3)=1.248$$
$$\vdots$$
$$x(\infty)=1.25$$

It was shown previously using the final value theorem

Delay Operator

Since $$X(z)=\sum_{n=0}^{\infty}x(n)z^{-n} \qquad x(n)=0 \quad n<0$$

The Z–transform of $x(n-k)$ is

$$Z[x(n-k)]=\sum_{n=0}^{\infty}[x(n-k)]z^{-n}$$

Let $m = n - k$

$$\Rightarrow Z[x(n-k)]=\sum_{m=-k}^{\infty}[x(m)]z^{-(m+k)} = \sum_{m=-k}^{\infty}[x(m)]z^{-m-k}$$

Since $x(n)$ is assumed 0 for $n<0$, then $x(m)=0$ for $m<0$

$$\Rightarrow Z[x(n-k)]=\sum_{m=0}^{\infty}[x(m)]z^{-m-k} = z^{-k}\sum_{m=0}^{\infty}[x(m)]z^{-m}$$

Since $x(nT)$ is assumed 0 for $n < 0$, then $x(mT) = 0$ for $m < 0$

$$Z[x(n-k)] = \sum_{m=-k}^{\infty} [x(m)] z^{-m-k)} = \sum_{m=0}^{\infty} [x(m)] z^{-m-k}$$

$$= z^{-k} \sum_{m=0}^{\infty} [x(m)] z^{-m} \quad = z^{-k} X(z)$$

Therefore ,delay by k samples periods is equivalent to multiplication by z^{-k}

Note , the similarity in Laplace domain,

$$x(t-t_0)u(t-t_0) \Leftrightarrow X(s)e^{-st_0}$$

Example 8-10

In this example we will show the effect of multiplying $X(z)$ by z^{-2}

From Example 8-9, we have ,

$$X(z) = \frac{z^2}{(z-1)(z-0.2)} \Rightarrow \begin{aligned} x(0) &= 1 \\ x(1) &= 1.2 \\ x(2) &= 1.24 \\ x(3) &= 1.248 \\ &\vdots \\ x(\infty) &= 1.25 \end{aligned}$$

Now let $Y(z) = z^{-2} X(z) = \dfrac{1}{(z-1)(z-0.2)}$

$$Y(z) = \frac{1}{(z-1)(z-0.2)}$$

$$\frac{Y(z)}{z} = \frac{1}{z(z-1)(z-0.2)} = \frac{5}{z} + \frac{1.25}{z-1} - \frac{6.25}{z-0.2}$$

$$\Rightarrow Y(z) = 5 + \frac{1.25z}{z-1} - \frac{6.25z}{z-0.2}$$

$$y(nT) = 5\delta(nT) + 1.25 - 6.25(0.2)^n$$

$$\begin{aligned} y(0) &= 0 \\ y(1) &= 0 \\ y(2) &= 1 \\ y(3) &= 1.2 \\ y(4) &= 1.24 \end{aligned}$$

Since

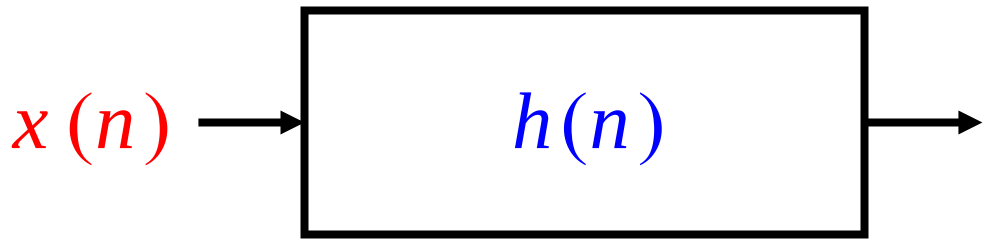
$$\begin{aligned} x(0) &= 1 \\ x(1) &= 1.2 \\ x(2) &= 1.24 \\ x(3) &= 1.248 \end{aligned}$$

$$\Rightarrow \mathrm{y}(n) = \mathrm{x}(n-2)$$

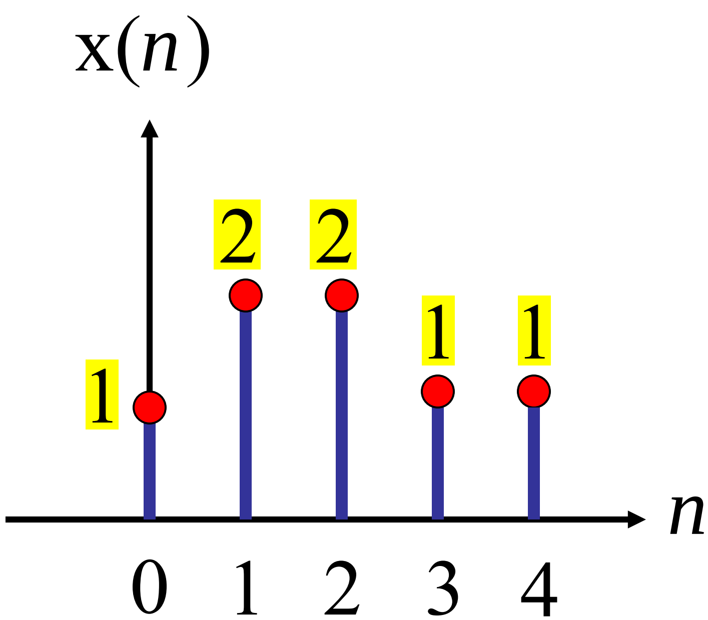
multiplying $X(z)$ by z^{-2} lead to delay by *2T*

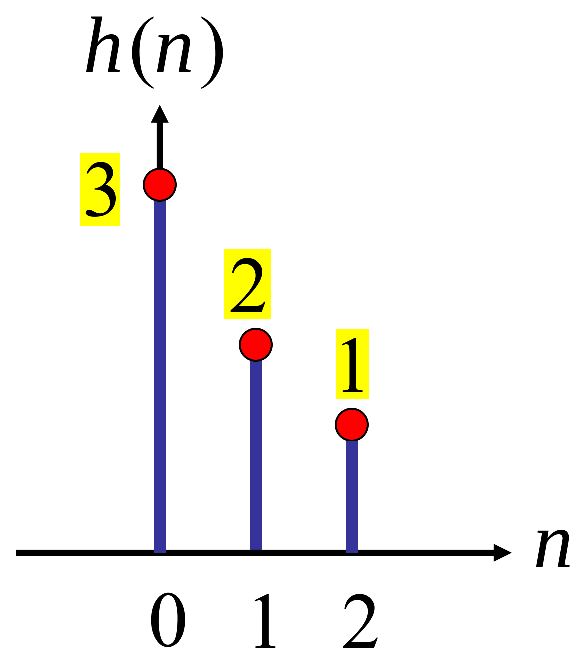
Example 8-12 **(Discrete Convolution)**

Let the input to a discrete-time system and the unit impulse response

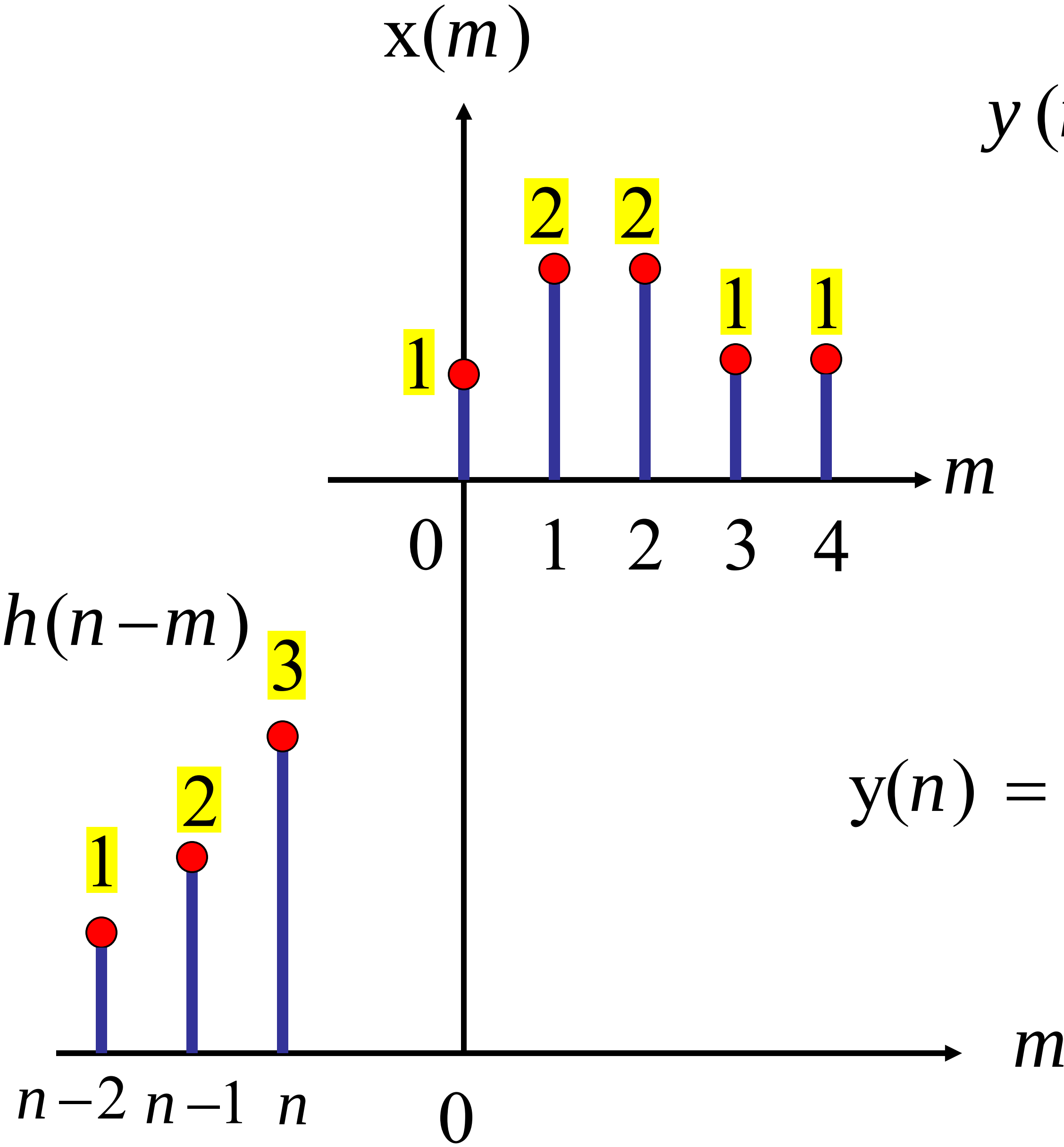


$$y(n) = \sum_{m=-\infty}^{\infty} x(m)h(n-m)$$



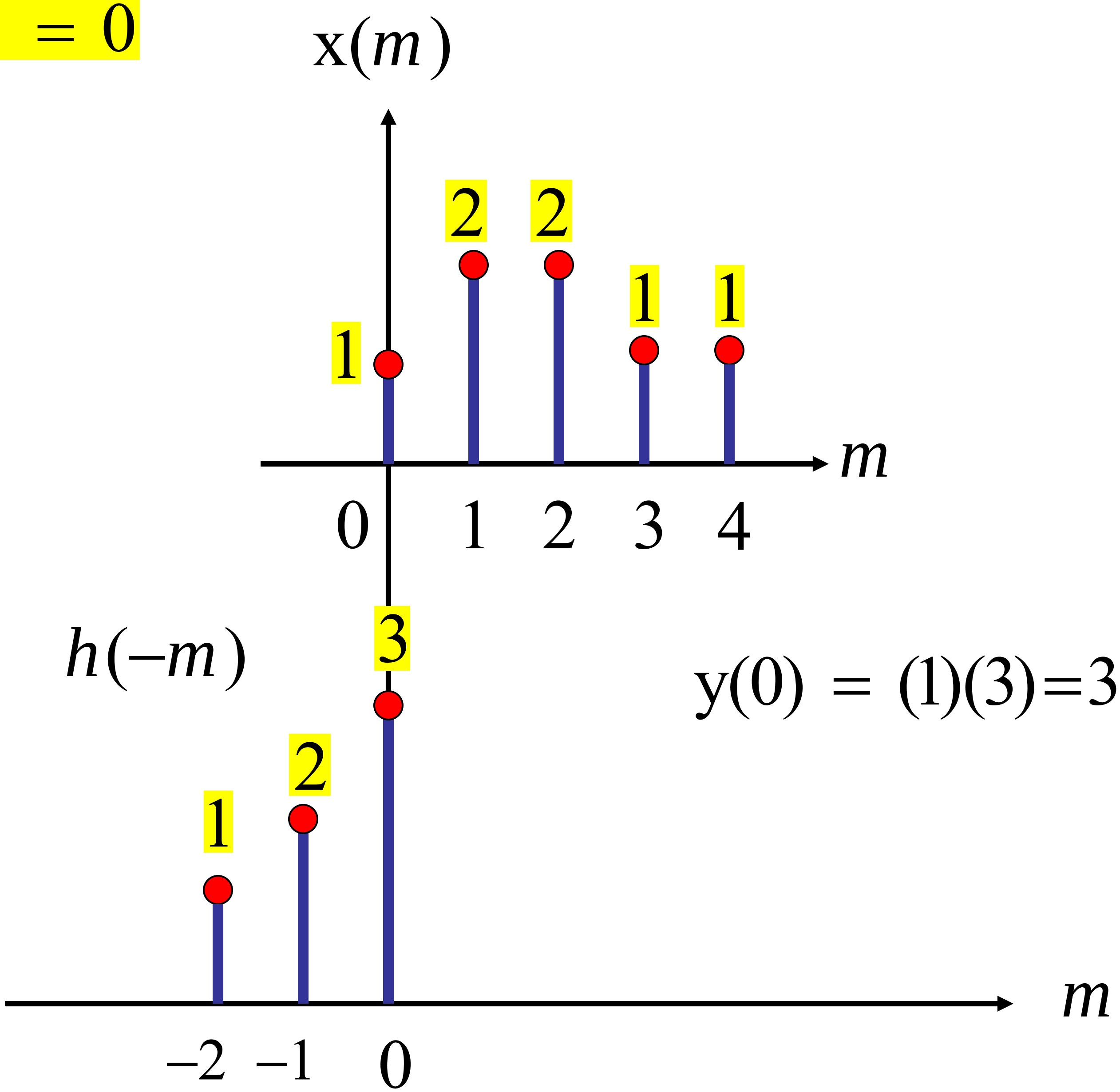


$$y(n)=\sum_{m=-\infty}^{\infty} x(m)h(n-m)$$



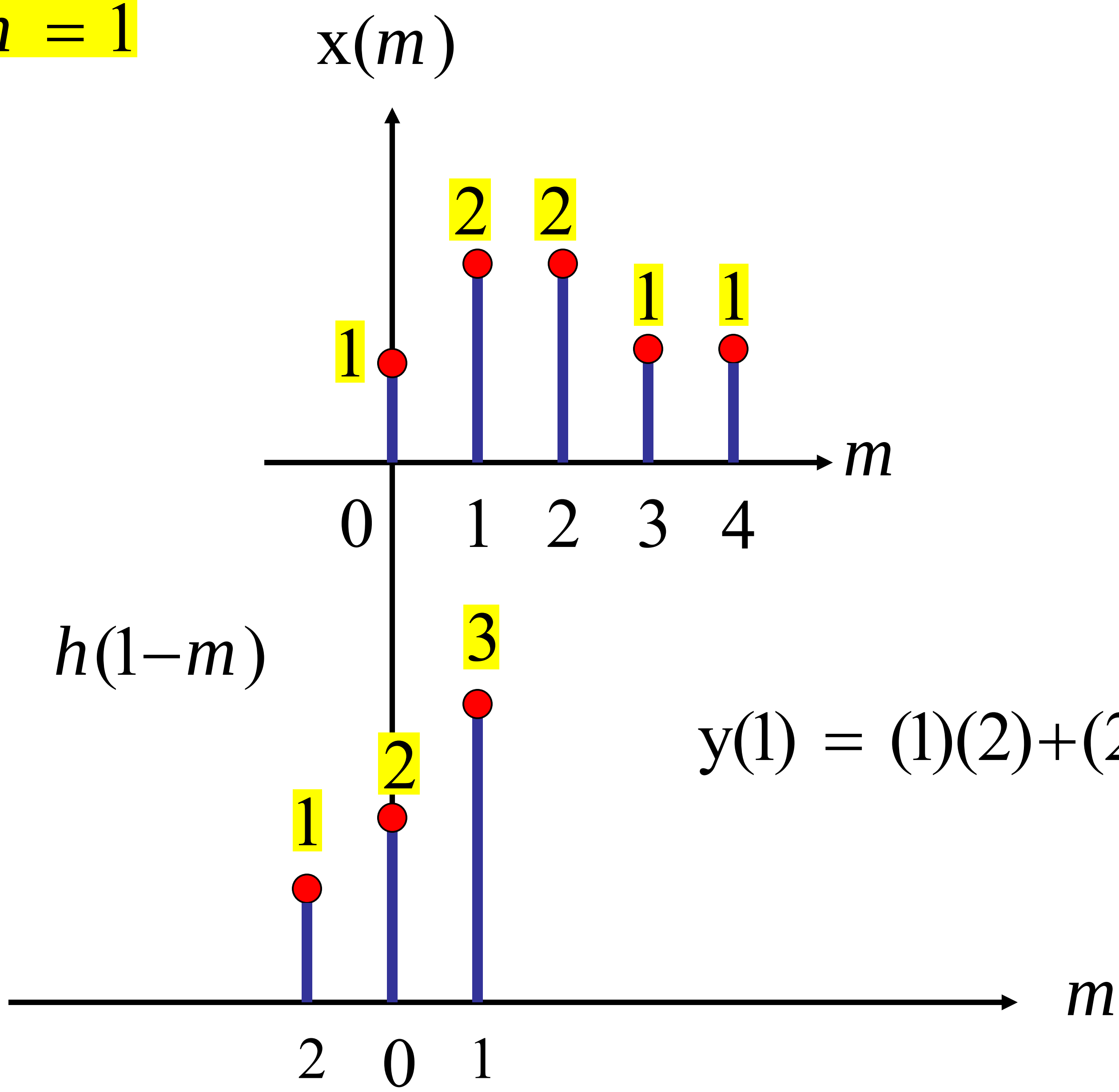
$$y(n) = 0 \qquad n < 0$$

$n = 0$



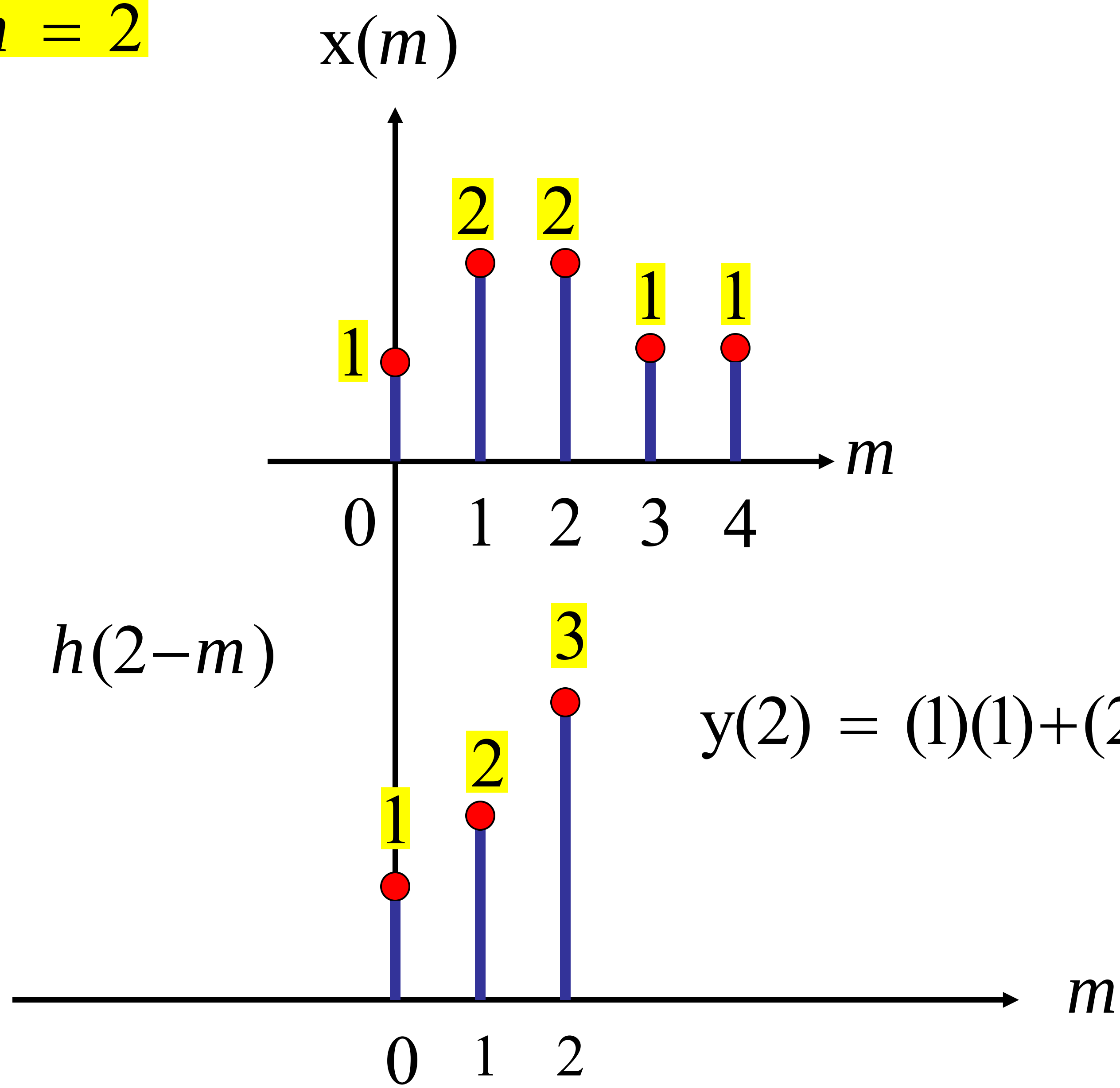
$y(0) = (1)(3) = 3$

$n = 1$



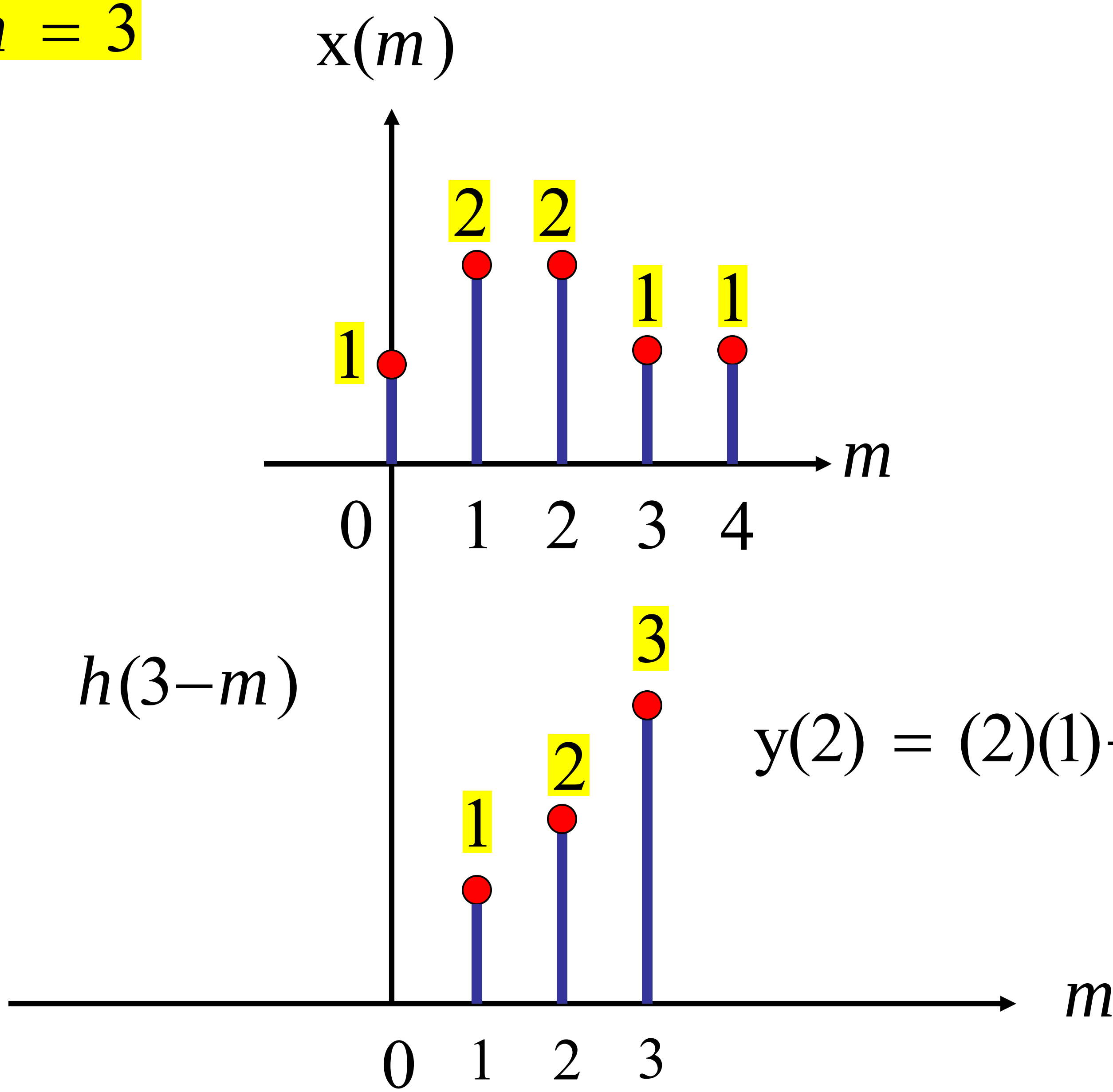
$y(1) = (1)(2)+(2)(3)=8$

$n = 2$



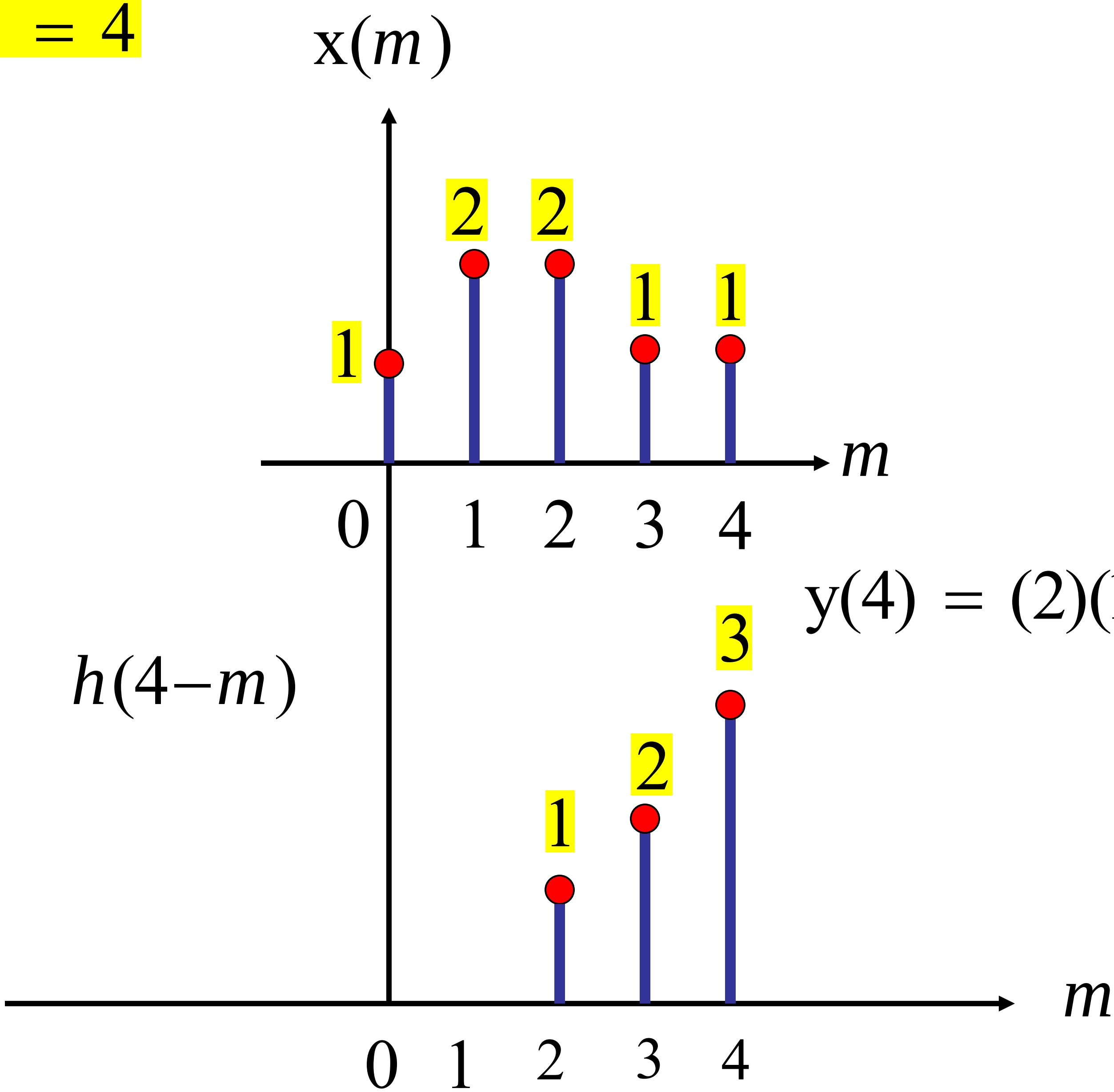
$$y(2) = (1)(1)+(2)(2)+(2)(3)=11$$

$n = 3$

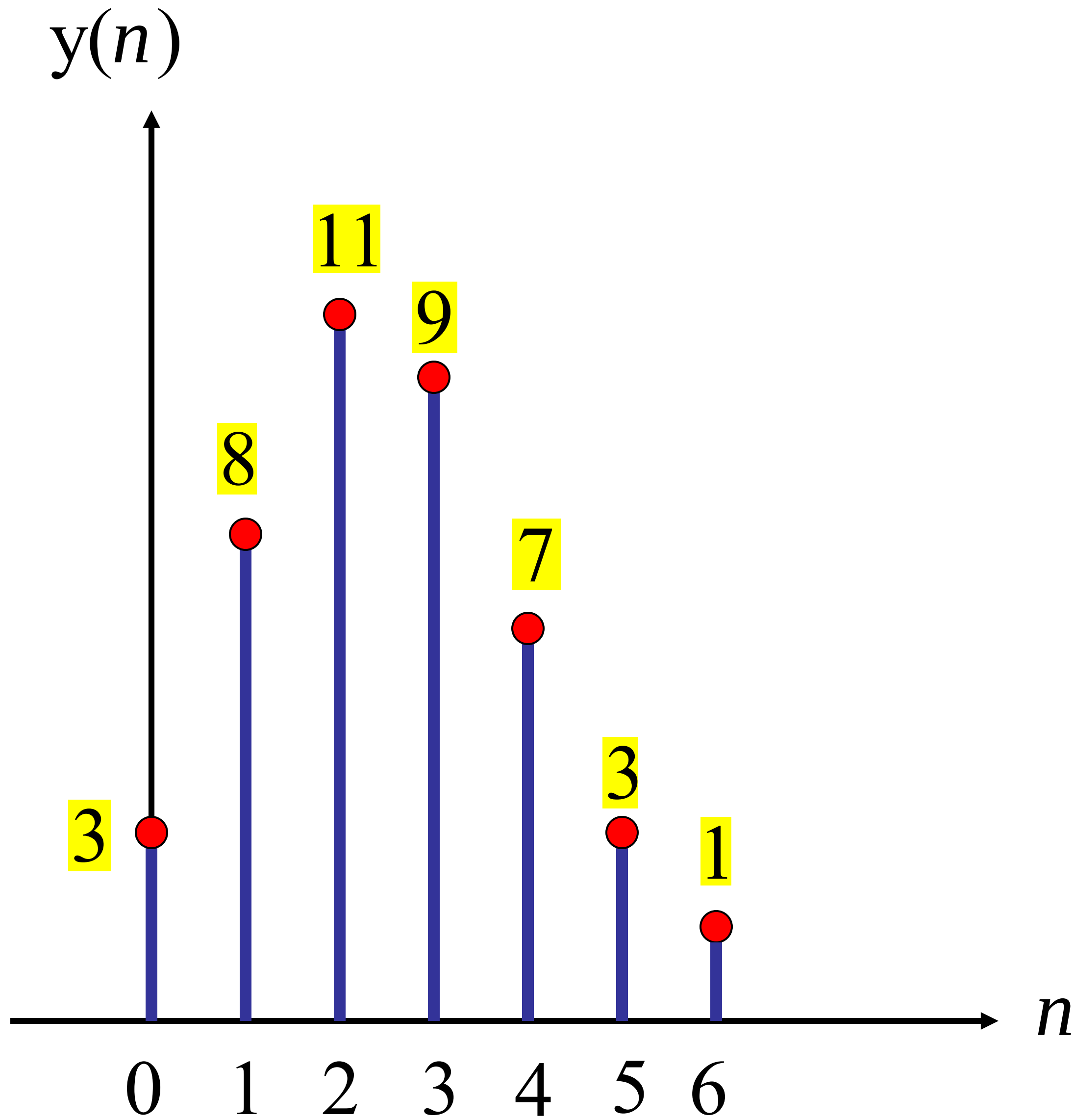


$$y(2) = (2)(1)+(2)(2)+(1)(3)=9$$

$n = 4$



$$y(4) = (2)(1)+(1)(2)+(1)(3)=7$$



y(n)
11
9
8
7
3
3
1
n
0
1
2
3
4
5
6

Example 8-13 $x(n)=\left(\frac{1}{2}\right)^{n}u(n) \qquad h(n)=\left(\frac{1}{3}\right)^{n}u(n)$

$$y(n)=\sum_{m=-\infty}^{\infty}x(m)h(n-m)=\sum_{m=-\infty}^{\infty}\left(\frac{1}{2}\right)^{m}u(m)\left(\frac{1}{3}\right)^{n-m}u(n-m)$$

Since $u(m)=0$ for $m<0$ and since $\mathrm{u}(n-m)=0$ for $m>n$

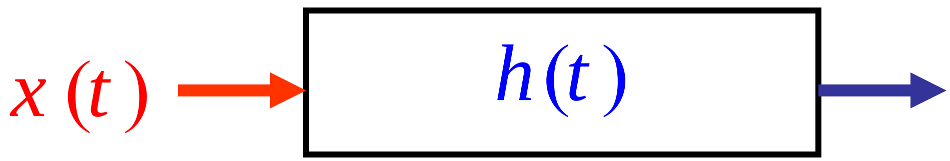
$$y(n)=\sum_{m=0}^{n}\left(\frac{1}{2}\right)^{m}\left(\frac{1}{3}\right)^{n-m}=\sum_{m=0}^{n}\left(\frac{1}{2}\right)^{m}\left(\frac{1}{3}\right)^{-m}\left(\frac{1}{3}\right)^{n}=\left(\frac{1}{3}\right)^{n}\sum_{m=0}^{n}\left(\frac{3}{2}\right)^{m}$$

Using the summation formula $\boxed{\sum_{m=0}^{N-1}x^{n}=\frac{1-x^{N}}{1-x}}$

$$y(n)=\left(\frac{1}{3}\right)^{n}\frac{1-(3/2)^{n+1}}{1-\frac{3}{2}}=3\left(\frac{1}{2}\right)^{n}-2\left(\frac{1}{3}\right)^{n} \quad n\geq 0$$

Stability

Recall from Chapter 2



$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} h(\lambda) x(t-\lambda) d\lambda$$

For the system to be **Bounded Input Bounded Output (BIBO)**

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

Now for discrete-time system

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = \sum_{m=-\infty}^{\infty} x(m)h(n-m)$$

For the system to be **Bounded Input Bounded Output (BIBO)**

$$\text{If} \quad |x(n)| < \infty \qquad \text{then} \quad |y(n)| < \infty$$

$$|y(n)| = \left| \sum_{m=-\infty}^{\infty} x(m)h(n-m) \right| \leq \sum_{m=-\infty}^{\infty} |x(m)h(n-m)|$$

for a bounded input $|x(n)| \leq M < \infty$ for all n

$$|y(n)| \leq \sum_{m=-\infty}^{\infty} M\,|h(n-m)| = M \sum_{m=-\infty}^{\infty} |h(n-m)|$$

Thus the system output is bounded if $\displaystyle\sum_{m=-\infty}^{\infty} |h(n-m)| < \infty$

Example 8-14

$$h(n)=\left[4\left(\frac{1}{3}\right)^n-3\left(\frac{1}{4}\right)^n\right]u(n)$$

Is The system $h(n)$ is **Bounded Input Bounded Output (BIBO)**

Since $h(n)\geq 0$ ⇒ $|h(n)|=h(n)$

$$\sum_{n=-\infty}^{\infty}\left|h(n-m)\right| = \sum_{n=0}^{\infty}\left[4\left(\frac{1}{3}\right)^n-3\left(\frac{1}{4}\right)^n\right] = \sum_{n=0}^{\infty}4\left(\frac{1}{3}\right)^n-\sum_{n=0}^{\infty}3\left(\frac{1}{4}\right)^n$$

$$=\frac{4}{1-\frac{1}{3}}-\frac{3}{1-\frac{1}{4}} = 2 \quad < \infty$$

⇒ The system is **Bounded Input Bounded Output (BIBO)**

For a continuous-time system

$$x(t) \longrightarrow \boxed{\quad h(t) \quad} \longrightarrow y(t)$$

A differential equation can be used to model a continuous-time system

$$a_n \frac{dy^{n}(t)}{dt^{n}} + a_{n-1}\frac{dy^{n-1}(t)}{dt^{n-1}} + \cdots + a_0 y(t) = b_m \frac{dx^{m}(t)}{dt^{m}} + b_{m-1}\frac{dx^{m-1}(t)}{dt^{m-1}} + \cdots + b_0 y(t)$$

Now if we take the Laplace Transform of both side (Assuming Zero initial Conditions)

$$a_n s^{n} Y(s) + a_{n-1} s^{n-1} Y(s) + \cdots + a_0 Y(s) = b_m s^{m} X(s) + b_{m-1} s^{m-1} X(s) + \cdots + b_0 X(s)$$

The transfer function $H(s)$ defined,

$$H(s) \triangleq \left.\frac{Y(s)}{X(s)}\right|_{\substack{\text{all initial} \\ \text{conditions} \\ \text{are zero}}} = \frac{b_m s^{m} + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^{n} + a_{n-1} s^{n-1} + \cdots + a_0}$$

For a discrete-time system

$$x(n) \longrightarrow \boxed{\;h(n)\;} \longrightarrow y(n)$$

A difference equation can be used to model a discrete-time system

$$y(n) + K_1 y(n-1) + K_2 y(n-2) + \cdots + K_m y(n-m)$$
$$= L_0 x(n) + L_1 x(n-1) + L_2 x(n-2) + \cdots + L_r x(n-r)$$

Now if we take the Z– Transform of both side

$$Y(z) + K_1 z^{-1} Y(z) + K_2 z^{-2} Y(z) + \cdots + K_m z^{-m} Y(z)$$
$$= L_0 X(z) + L_1 z^{-1} X(z) + L_2 z^{-2} X(z) + \cdots + L_r z^{-r} X(z)$$

The transfer function $H(z)$ defined,

$$H(z) \triangleq \frac{Y(z)}{X(z)} = \frac{L_0 + L_1 z^{-1} + L_2 z^{-2} + \cdots + L_r z^{-r}}{1 + K_1 z^{-1} + K_2 z^{-2} + \cdots + K_m z^{-m}}$$

For a discrete-time system

$$x(n) \longrightarrow \boxed{\quad h(n) \quad} \longrightarrow y(n) = \sum_{m=-\infty}^{\infty} x(m)h(n-m)$$

$$X(z) \qquad\qquad H(z) \qquad\qquad Y(z) = X(z)H(z)$$

From Example 8-13

$$x(n) = \left(\frac{1}{2}\right)^n u(n) \qquad h(n) = \left(\frac{1}{3}\right)^n u(n)$$

Using discrete convolution

$$y(n) = \sum_{m=-\infty}^{\infty} x(m)h(n-m) = 3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n \quad n \geq 0$$

Now we will find y(n) using the Z-transform

$$x(n) \longrightarrow \boxed{\quad h(n) \quad} \longrightarrow y(n)$$

$$x(n) = \left(\frac{1}{2}\right)^n u(n) \qquad h(n) = \left(\frac{1}{3}\right)^n u(n)$$

$$X(z) = \frac{z}{z-(1/2)} \qquad H(z) = \frac{z}{z-(1/3)}$$

$$\Rightarrow \quad Y(z) = X(z)H(z) = \left(\frac{z}{z-(1/2)}\right)\left(\frac{z}{z-(1/3)}\right)$$

$$\frac{Y(z)}{z} = \frac{z}{(z-(1/2))(z-(1/3))} = \frac{3}{(z-(1/2))} - \frac{2}{(z-(1/3))}$$

$$X(z) \longrightarrow \boxed{\quad H(z) \quad} \longrightarrow Y(z)$$

$$\frac{Y(z)}{z} = \frac{3}{(z-(1/2))} - \frac{2}{(z-(1/3))}$$

$$Y(z) = \frac{3z}{(z-(1/2))} - \frac{2z}{(z-(1/3))}$$

$$\Rightarrow \quad y(n) = 3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n \quad n \geq 0$$