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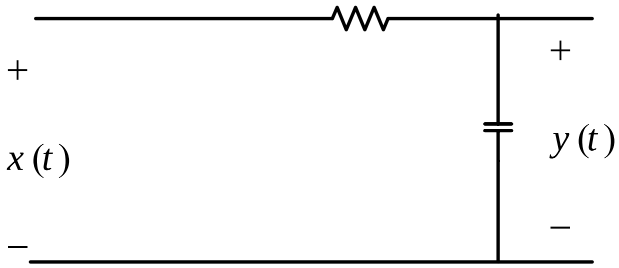
Chapter 5 Laplace Transform

Chapter 5 The Laplace Transform

Consider the following RC circuit (**System**)



System analysis in the time domain involves (finding y(t)):

R

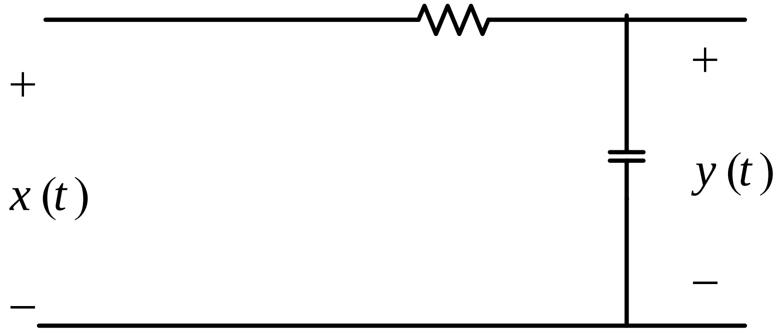
Solving the differential equation $$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

OR

C

Using the convolution integral $$y(t) = x(t)*h(t)$$

Both Techniques can results in tedious (ممل)mathematical operation



$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

R

Fourier Transform provided an alternative approach

Differential Equation → Algebraic Equation ***C***

$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

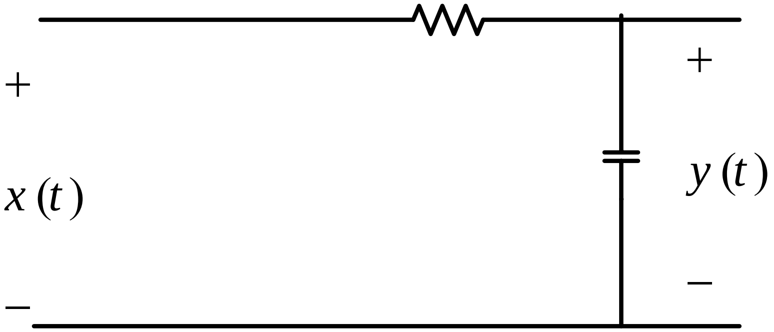
$$RC(j2\pi f)Y(\mathrm{f})+Y(\mathrm{f})=X(\mathrm{f})$$

solve for $Y(\mathrm{f})$

$$Y(\mathrm{f}) = \frac{X(\mathrm{f})}{\left[(j2\pi fRC) + 1\right]}$$

Inverse Back

$y(t)$



$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

$$RC(j2\pi f)Y(f)+Y(f)=X(f)$$

R

$$Y(f) = \frac{X(f)}{[(j2\pi fRC)+1]}$$

Inverse Back

$y(t)$

C

Unfortunately , there are many signals of interest that arise in system analysis for which the Fourier Transform does not exists

A more general transform is needed

Fourier Transform pairs was defined

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \qquad X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

Now multiply $x(t)$ by $e^{-\sigma t}$ and takes the Fourier Transform

$$FT\left[e^{-\sigma t} x(t)\right] = \int_{0}^{\infty} x(t) e^{-\sigma t} e^{-j\omega t} dt = \int_{0}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt = X(\sigma + j\omega)$$

Let $s \triangleq \sigma + j\omega$ Complex Frequency

$$\Rightarrow \quad FT\left[e^{-\sigma t} x(t)\right] = \int_{0}^{\infty} x(t) e^{-st} dt = X(s) \triangleq \mathrm{L}\left[x(t)\right]$$

were $\mathrm{L}[\quad]$ Denotes the operation of obtaining the Laplace Transform

The unilateral (آحادي الجانب) Laplace Transform defined as

$$\mathrm{L}\left[x(t)\right] \triangleq \int_{0}^{\infty} x(t)e^{-st}dt = X(s)$$

The inverse Laplace Transform $\mathrm{L}^{-1}\left[\quad\right]$ can be obtained as follows:

Since

$$FT\left[e^{-\sigma t}x(t)\right] = \int_{0}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt = X(\sigma + j\omega)$$

$$x(t)e^{-\sigma t} = FT^{-1}[X(\sigma + j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega$$

$$\Rightarrow x(t) = e^{-\sigma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega$$

$$x(t) = e^{\sigma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega \quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{\sigma t}e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega$$

Change the variable of integration

$$s \triangleq \sigma + j\omega \Rightarrow ds = jd\omega \quad \text{OR} \quad d\omega = \frac{ds}{j}$$

The limits $\omega = \infty \rightarrow s = \sigma + j\infty$

$\omega = -\infty \rightarrow s = \sigma - j\infty$

$$\Rightarrow x(\mathrm{t}) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

5-2 Examples of Evaluating Laplace Transform

Let $x(\mathrm{t}) = 1$, then

$$X(s) = \int_0^\infty x(t)e^{-st}dt = \int_0^\infty (1)e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = \left.\frac{e^{-(\sigma+j\omega)t}}{-s}\right|_0^\infty$$

$$= \left.\frac{e^{-\sigma t}e^{-j\omega t}}{-s}\right|_0^\infty = -\left.\frac{e^{-\sigma t}}{s}[\cos\omega t - j\sin\omega t]\right|_0^\infty$$

$$= -\frac{\overset{\boxed{0 \text{ if } \sigma > 0}}{e^{-\sigma(\infty)}}}{s}\overbrace{[\cos\omega(\infty) - j\sin\omega(\infty)]}^{\boxed{\text{between } -1 \text{ and } 1}} + \frac{\overset{\boxed{1}}{e^{-\sigma(0)}}}{s}[\overset{\boxed{1}}{\cos\omega(0)} - j\overset{\boxed{0}}{\sin\omega(0)}]$$

$$= \frac{1}{s} \quad \boxed{\text{if } \sigma > 0 \quad \text{OR} \quad \mathrm{Re}(s) > 0}$$

Note if $\sigma < 0 \Rightarrow e^{-\sigma(\infty)} = \infty$
$\Rightarrow$ Solution doesn't exist

➡ $\mathrm{L}[\,1\,] = \dfrac{1}{s} \qquad \mathrm{Re}(s) > 0$

Example Let $x(t) = e^{-\alpha t}u(t)$

$$\mathrm{L}[\, e^{-\alpha t}u(t)\,] = \int_0^\infty e^{-\alpha t}e^{-st}dt = \int_0^\infty e^{-(\alpha+s)t}dt = -\left.\frac{e^{-(\alpha+s)t}}{(\alpha+s)}\right|_0^\infty$$

$$= -\left.\frac{e^{-(\alpha+\sigma+j\omega)t}}{(\alpha+s)}\right|_0^\infty = -\left.\frac{e^{-(\alpha+\sigma)t}e^{-j\omega t}}{(\alpha+s)}\right|_0^\infty = -\left.\frac{e^{-(\alpha+\sigma)t}}{(\alpha+s)}[\cos\omega t - j\sin\omega t]\right|_0^\infty$$

$$= -\frac{\overset{0 \ \text{if}\ (\alpha+\sigma)>0}{e^{-(\alpha+\sigma)(\infty)}}}{(\alpha+s)}\left[\overset{\text{between } -1 \text{ and } 1}{\cos\omega(\infty) - j\sin\omega(\infty)}\right] + \frac{\overset{1}{e^{-(\alpha+\sigma)(0)}}}{(\alpha+s)}\left[\overset{1}{\cos\omega(0)} - j\,\overset{0}{\sin\omega(0)}\right]$$

$$= \frac{1}{(\alpha+s)}$$

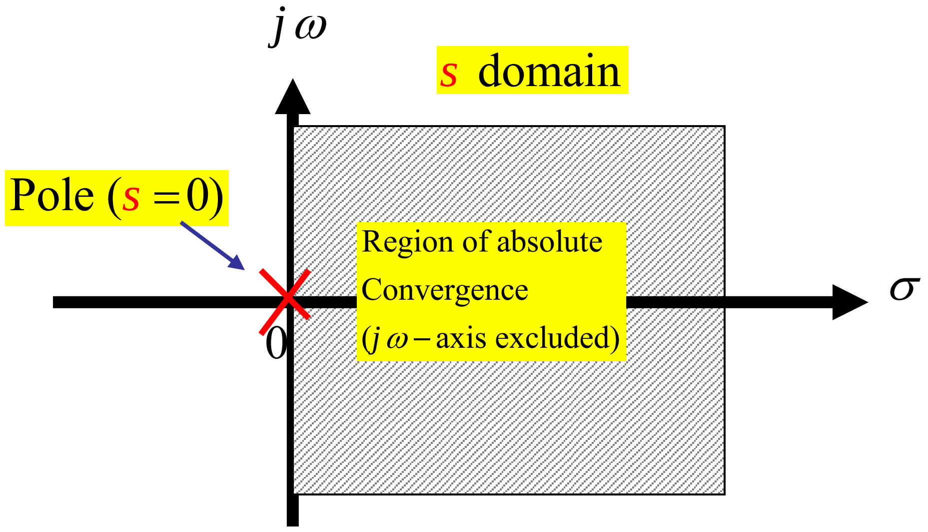
$$\Rightarrow \mathrm{L}[\, e^{-\alpha t}u(t)\,] = \frac{1}{(\alpha+s)} \quad \text{if} \quad \mathrm{Re}(\alpha+s) > 0$$

$$\text{OR} \quad \mathrm{Re}(s) > -\mathrm{Re}(\alpha)$$

Region of Convergence

$$L[\ 1\ ] = L[\ u(t)\ ] = \frac{1}{s} \qquad \mathrm{Re}(s) > 0$$

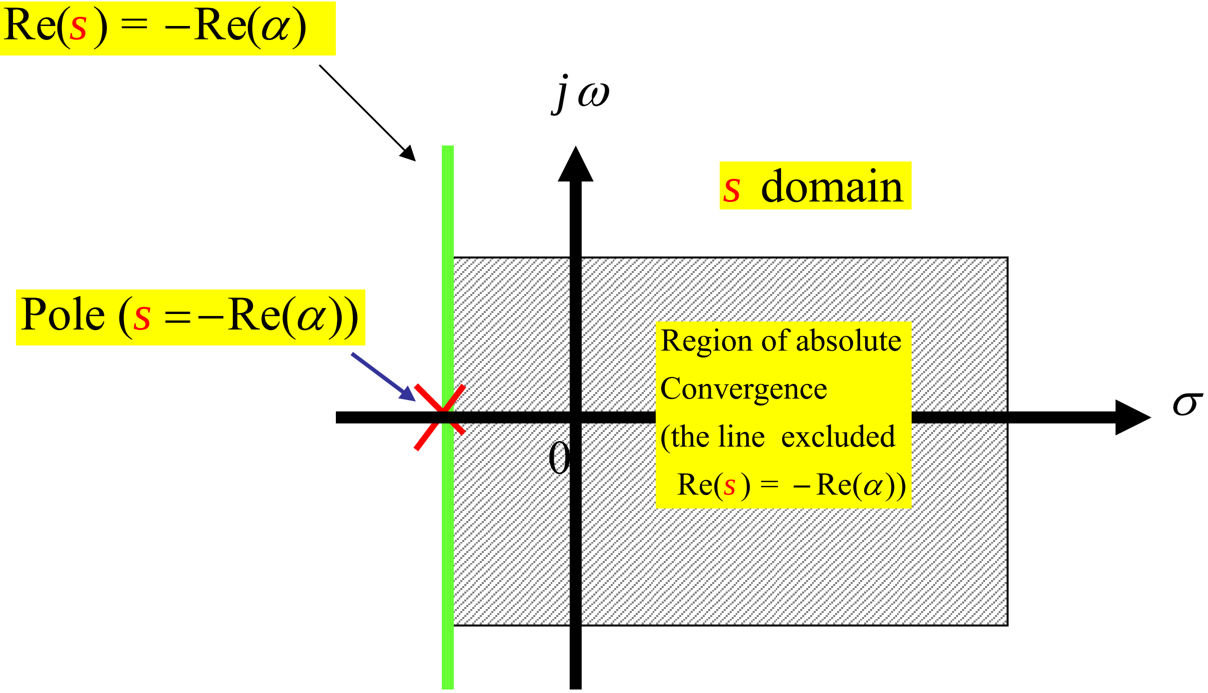
Since $s = \sigma + j\omega$, then s in general complex



A pole × is where the Laplace Transform $X(s) \rightarrow \infty$

$$L[\ e^{-\alpha t}u(t)\ ] = \frac{1}{(\alpha + s)} \qquad \text{if} \quad \mathrm{Re}(\alpha + s) > 0$$

$$\text{OR} \quad \mathrm{Re}(s) > -\mathrm{Re}(\alpha)$$



$\mathrm{Re}(s) = -\mathrm{Re}(\alpha)$

s domain

Pole ($s = -\mathrm{Re}(\alpha)$)

Region of absolute Convergence (the line excluded $\mathrm{Re}(s) = -\mathrm{Re}(\alpha)$)

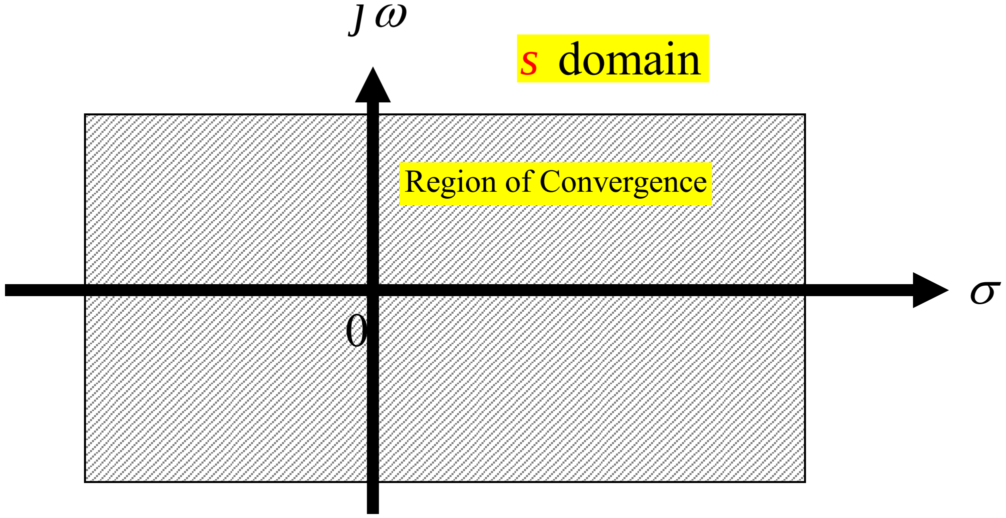
$$L[\ \delta(t)\ ] = \int_{0}^{\infty} \delta(t) e^{-st} dt$$

Assuming the lower limit is 0^{-}

$$L[\ \delta(t)\ ] = \int_{0^{-}}^{\infty} \delta(t) e^{-st} dt = e^{-st}\Big|_{t=0} = 1$$

The value of s make no difference

Region of Convergence



5-3 Some Laplace Transform Theorems

Theorem 1: Linearity

Let $\mathrm{L}[\,x_1(\mathrm{t})] = X_1(s)$ $\quad$ $\mathrm{L}[\,x_2(\mathrm{t})] = X_2(s)$

Then $\mathrm{L}[\,\mathrm{a}_1 x_1(\mathrm{t}) + \mathrm{a}_2 x_2(\mathrm{t})\,] = \mathrm{a}_1 X_1(s) + \mathrm{a}_2 X_2(s)$

Proof

$$\mathrm{L}[\,\mathrm{a}_1 x_1(\mathrm{t}) + \mathrm{a}_2 x_2(\mathrm{t})\,] = \int_0^\infty [\,\mathrm{a}_1 x_1(\mathrm{t}) + \mathrm{a}_2 x_2(\mathrm{t})\,] e^{-st}\,dt$$

$$= \int_0^\infty \mathrm{a}_1 x_1(\mathrm{t}) e^{-st}\,dt + \int_0^\infty \mathrm{a}_2 x_2(\mathrm{t}) e^{-st}\,dt \quad = \mathrm{a}_1 X_1(s) + \mathrm{a}_2 X_2(s)$$

Example

$$\mathrm{L}[\ \cos\omega_0 t\ ] = \mathrm{L}\left[\frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}\right] = \frac{1}{2}\mathrm{L}[e^{j\omega_0 t}] + \frac{1}{2}\mathrm{L}[e^{-j\omega_0 t}]$$

Since $\mathrm{L}[\ e^{-\alpha t}u(t)\ ] = \dfrac{1}{(\alpha + s)}$

Then $\mathrm{L}[e^{j\omega_0 t}] = \dfrac{1}{(j\omega_0 + s)}$ $\qquad \mathrm{L}[e^{-j\omega_0 t}] = \dfrac{1}{(-j\omega_0 + s)}$

$$\mathrm{L}[\ \cos\omega_0 t\ ] = \frac{1}{2}\frac{1}{(s + j\omega_0)} + \frac{1}{2}\frac{1}{(s - j\omega_0)} = \frac{s}{s^2 + \omega_0^2}$$

Similarly $\mathrm{L}[\ \sin\omega_0 t\ ] = \dfrac{\omega_0}{s^2 + \omega_0^2}$

Theorem 2: Transform of Derivatives

$$\mathrm{L}\left[\frac{dx(t)}{dt}\right] = sX(s) - x(0^-)$$

Proof

$$\mathrm{L}\left[\frac{dx(t)}{dt}\right] = \int_0^\infty \left[\frac{dx(t)}{dt}\right] e^{-st}\, dt$$

Integrating by parts, $u = e^{-st} \quad dv(t) = dx(t)$

$$\Rightarrow \quad du = -se^{-st} \quad v(t) = x(t)$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$u = e^{-st} \quad dv(t) = dx(t) \qquad du = -se^{-st} \quad v(t) = x(t)$$

$$\mathrm{L}\left[\frac{dx(t)}{dt}\right] = \int_0^\infty \left[\frac{dx(t)}{dt}\right] e^{-st}\,dt = e^{-st}x(t)\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} x(t)e^{-st}\,dt$$

$$= \left[\overset{\nearrow\, 0}{e^{-s(\infty)}}x(\infty) - \overset{\nearrow\, 1}{e^{-s(0)}}x(0^-)\right] + sX(s) = sX(s) - x(0^-)$$

$$\Rightarrow \quad \frac{dx(t)}{dt} \Leftrightarrow sX(s) - x(0^-)$$

$$\frac{dx(t)}{dt} \Leftrightarrow sX(s) - x(0^-)$$

$$\frac{d^2x(t)}{dt^2} \Leftrightarrow s^2X(s) - s\,x(t)\Big|_{t=0} - \frac{dx(t)}{dt}\Big|_{t=0}$$

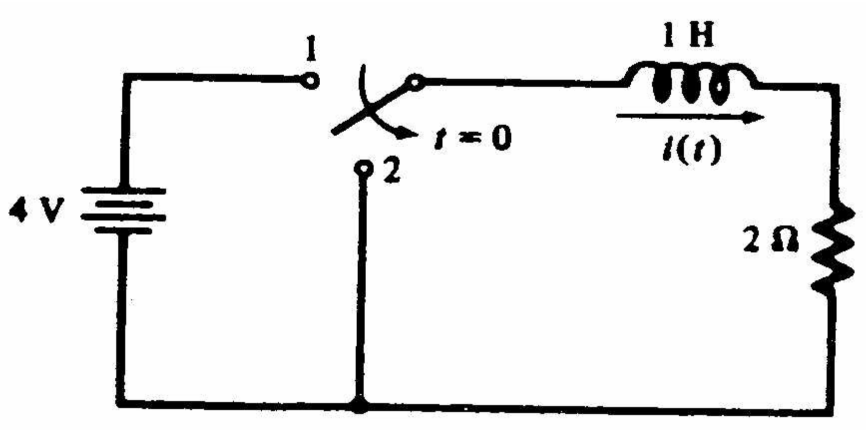
$$\frac{d^3x(t)}{dt^3} \Leftrightarrow s^3X(s) - s^2x(t)\Big|_{t=0} - s\frac{dx(t)}{dt}\Big|_{t=0} - \frac{d^2x(t)}{dt^2}\Big|_{t=0}$$

$$\frac{d^nx(t)}{dt^n} \Leftrightarrow s^nX(s) - s^{n-1}x(t)\Big|_{t=0} -$$

$$s^{n-2}\frac{dx(t)}{dt}\Big|_{t=0} - s^{n-3}\frac{d^2x(t)}{dt^2}\Big|_{t=0} - \cdots - \frac{d^{n-1}x(t)}{dt^{n-1}}\Big|_{t=0}$$

EXAMPLE 5-2

Consider the circuit shown



$$\frac{di(t)}{dt} + 2i(t) = \begin{cases} 4, & t \le 0 \\ 0, & t > 0 \end{cases}$$

Taking the Laplace transform of both sides starting at $t = 0^-$

$$sI(s) - i(0^-) + 2I(s) = 0$$

Assuming that the circuit was in steady state for $t < 0$. $\Rightarrow$ $i(0^-) = \frac{4}{2} = 2$

$\Rightarrow$ $I(s)(s + 2) - 2 = 0$ $\Rightarrow$ $I(s) = \frac{2}{s + 2}$ $\Rightarrow$ $i(t) = 2e^{-2t}u(t)$

$\Rightarrow$ $i(t) = \begin{cases} 2e^{-2t}, & t > 0 \\ 2, & t \le 0 \end{cases}$

Theorem 3: Laplace Transform of an Integral

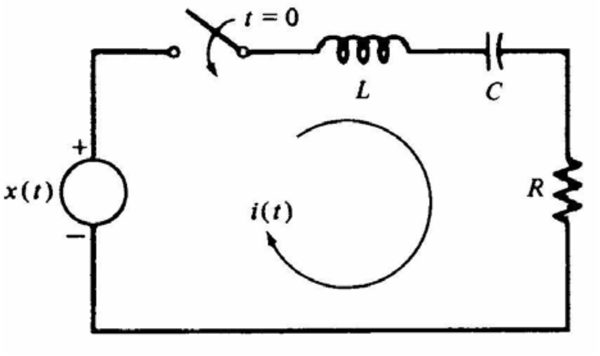
Let $$y(t)=\int_{-\infty}^{t} x(\lambda)d\lambda$$

Then $$\mathrm{L}\left[\int_{-\infty}^{t} \mathrm{x}(\lambda)d\lambda\right] = \frac{X(s)}{s} + \frac{y(0^{-})}{s}$$

were $$y(0^{-})=\int_{-\infty}^{0} x(\lambda)d\lambda\Big|_{t=0^{-}}$$

Proof see the book

EXAMPLE 5-3



$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(\lambda)\,d\lambda = x(t)$$

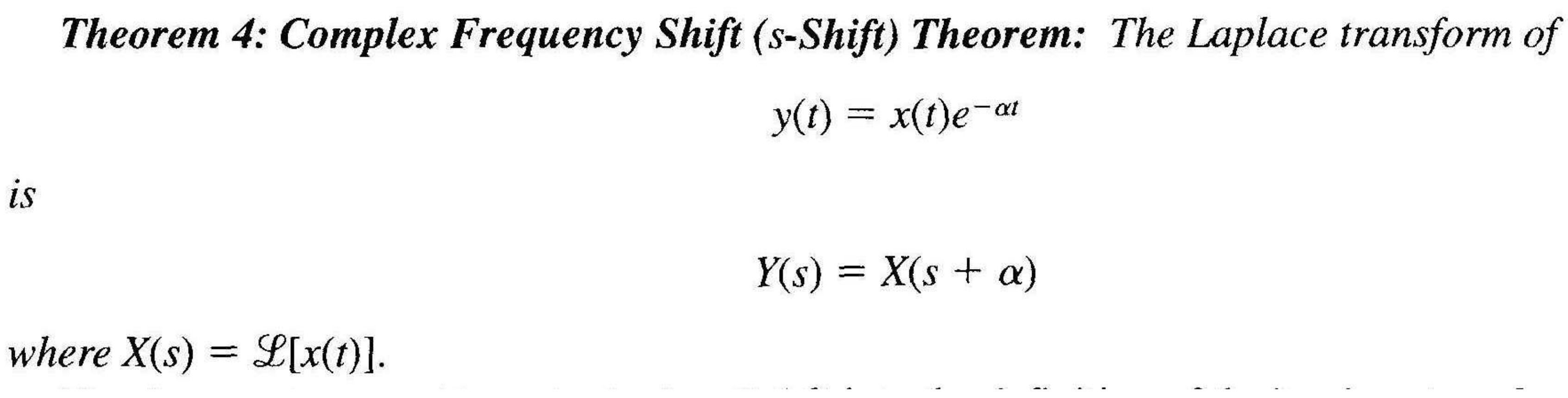
$$i(0^{-}) = 0$$

$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i(\lambda)d\lambda \qquad \Rightarrow \qquad v_C(0^{-}) = \frac{1}{C}\int_{-\infty}^{0^{-}} i(\lambda)d\lambda$$

$$LsI(s) + RI(s) + \frac{I(s)}{sC} + \frac{v_c(0^{-})}{s} = X(s)$$

Solving for $I(s)$

$$I(s) = \frac{sX(s) - v_c(0^{-})}{L[s^2 + (R/L)s + 1/LC]}$$



Theorem 4: Complex Frequency Shift (s-Shift) Theorem: *The Laplace transform of*

$$y(t) = x(t)e^{-\alpha t}$$

is

$$Y(s) = X(s + \alpha)$$

where $X(s) = \mathcal{L}[x(t)]$.

Proof see the book (similar to the Fourier Transform Property)

$$\mathrm{L}[\ \cos\omega_0 t\ ] = \frac{s}{s^2+\omega_0^2}$$

$$\mathscr{L}[e^{-\alpha t}\cos\omega_0 t] = \frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$$

$$\mathrm{L}[\ \sin\omega_0 t\ ] = \frac{\omega_0}{s^2+\omega_0^2}$$

$$\mathscr{L}[e^{-\alpha t}\sin\omega_0 t] = \frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$$

Theorem 2: Delay Theorem

$$\mathrm{L}\left[x(t-t_0)u(t-t_0)\right] = e^{-st_0}X(s) \qquad t_0 > 0$$

Proof $$\mathrm{L}\left[x(t-t_0)u(t-t_0)\right] = \int_{t_0}^{\infty} x(t-t_0)e^{-st}dt$$

Let $t' = t - t_0 \quad \Rightarrow \quad dt = dt'$

Limits $t : t_0 \rightarrow \infty$

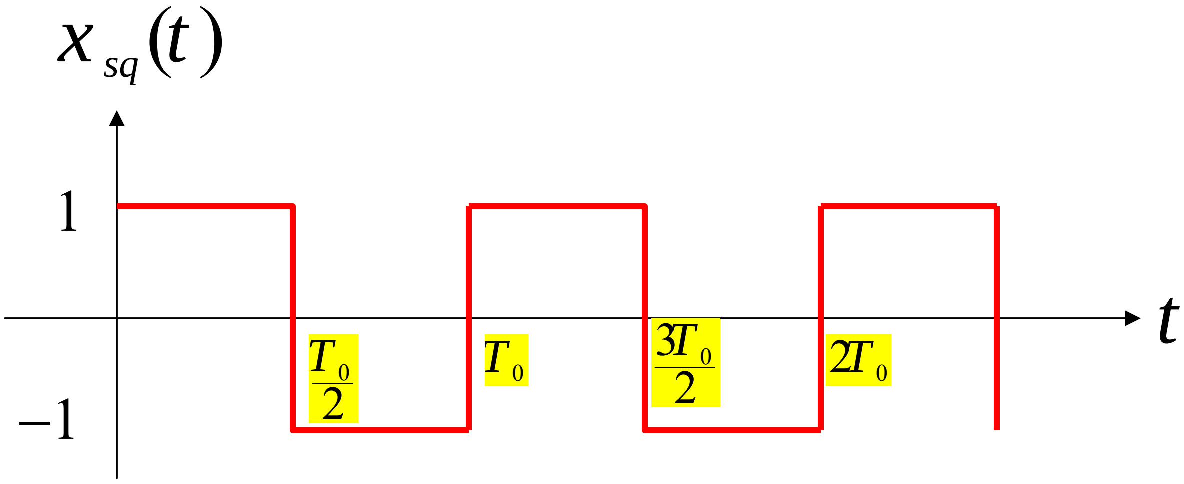
$t' : 0 \rightarrow \infty$

$$\mathrm{L}\left[x(t-t_0)u(t-t_0)\right] = \int_{0}^{\infty} x(t')e^{-s(t'+t_0)}dt' = e^{-st_0}\int_{0}^{\infty} x(t')e^{-st'}dt' = X(s)e^{-st_0}$$

Note $u(t-t_0)$ is necessary to give proper limit

$t_0 > 0$ (shift right) is necessary to give proper limit since Laplace will not include the portion of $x(t-t_0)u(t-t_0)$ for t < 0

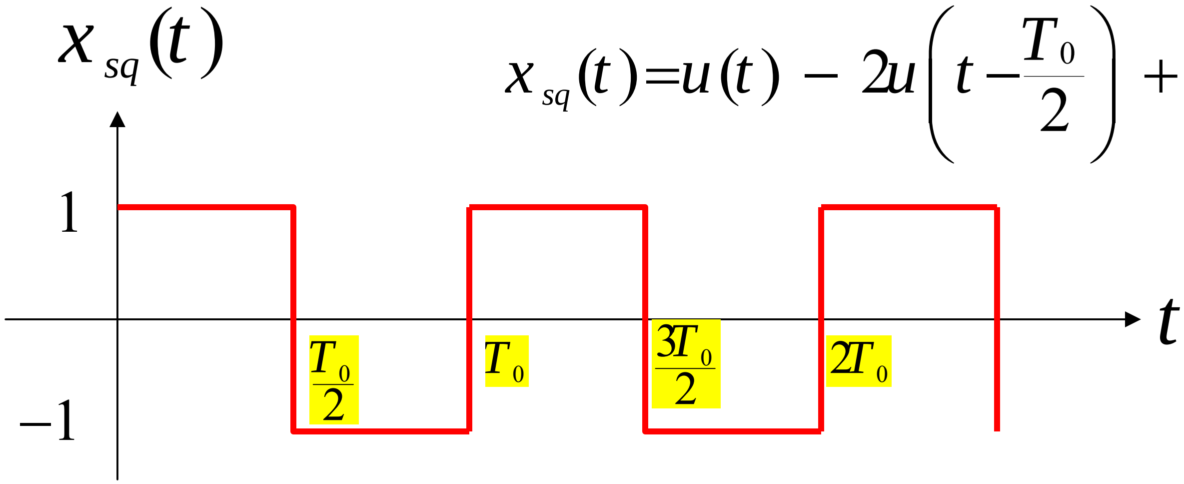
Example 5-5 Let $x_{sq}(t)$ be the square wave beginning at $t = t_0$



$$x_{sq}(t) = u(t) - 2u\left(t - \frac{T_0}{2}\right) + 2u(t - T_0) - \cdots$$

$$L\left[x_{sq}(t)\right] = L\left[u(t) - 2u\left(t - \frac{T_0}{2}\right) + 2u(t - T_0) - \cdots\right]$$

$$L\left[x_{sq}(t)\right] = L\left[u(t)\right] - 2L\left[u\left(t - \frac{T_0}{2}\right)\right] + 2L\left[u(t - T_0)\right] - \cdots$$



$$x_{sq}(t) = u(t) - 2u\left(t - \frac{T_0}{2}\right) + 2u(t - T_0) - \cdots$$

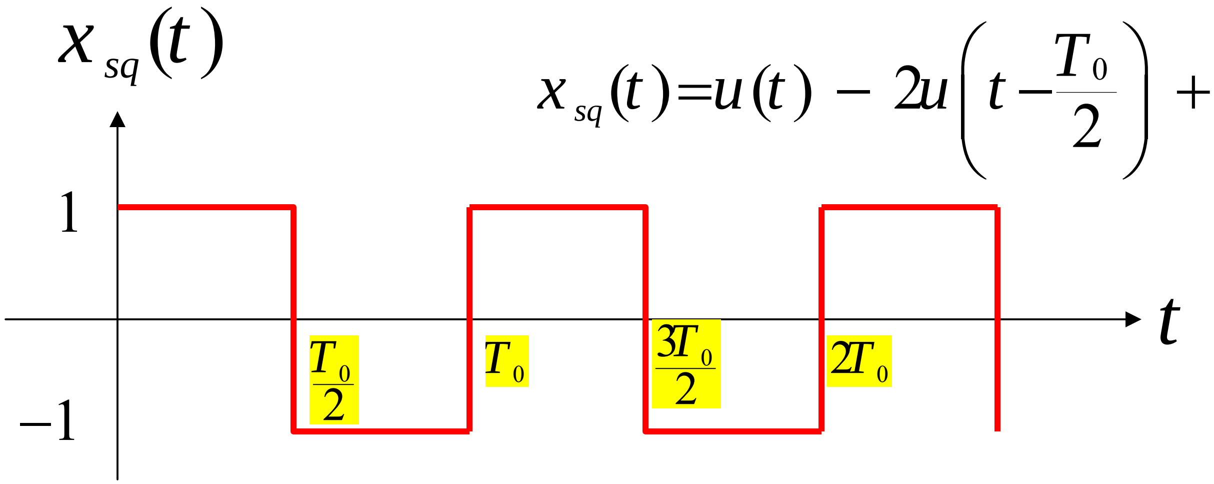
$$L\left[x_{sq}(t)\right] = L\left[u(t)\right] - 2L\left[u\left(t - \frac{T_0}{2}\right)\right] + 2L\left[u(t - T_0)\right] - \cdots$$

$$\Big\downarrow \qquad\qquad \Big\downarrow \qquad\qquad \Big\downarrow$$

$$\frac{1}{s} \qquad 2e^{-s\frac{T_0}{2}}\frac{1}{s} \qquad 2e^{-sT_0}\frac{1}{s}$$

$$L\left[x_{sq}(t)\right] = \frac{1}{s} - 2e^{-s\frac{T_0}{2}}\frac{1}{s} + 2e^{-sT_0}\frac{1}{s} - \cdots$$

$$= \frac{1}{s}\left(1 - 2e^{-s\frac{T_0}{2}} + 2e^{-sT_0} - \cdots\right)$$



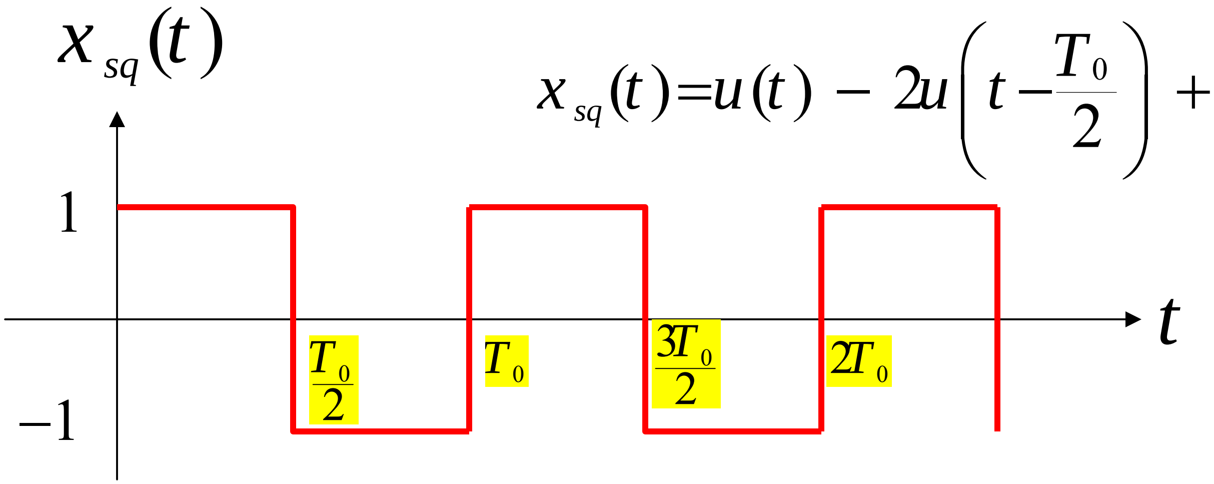
$$x_{sq}(t)=u(t)-2u\left(t-\frac{T_0}{2}\right)+2u(t-T_0)-\cdots$$

$$L\left[x_{sq}(t)\right]=\frac{1}{s}\left(1-2e^{-s\frac{T_0}{2}}+2e^{-sT_0}-\cdots\right)$$

Since $\frac{1}{1+x}=1-x+x^2-x^3 \qquad |x|<1$

$$1-2e^{-s\frac{T_0}{2}}+2e^{-sT_0}-\cdots=2\left(1-2e^{-s\frac{T_0}{2}}+2e^{-sT_0}-\cdots\right)-1$$

$$=2\left(1-2e^{-s\frac{T_0}{2}}+2e^{-s\left(\frac{T_0}{2}\right)^2}-\cdots\right)-1=2\left(\frac{1}{1+e^{-s\frac{T_0}{2}}}\right)-1=\left(\frac{2}{1+e^{-s\frac{T_0}{2}}}\right)-1$$



$$x_{sq}(t)=u(t)-2u\left(t-\frac{T_0}{2}\right)+2u(t-T_0)-\cdots$$

$$L\left[x_{sq}(t)\right]=\frac{1}{s}\left(1-2e^{-s\frac{T_0}{2}}+2e^{-sT_0}-\cdots\right)=\frac{1}{s}\left(\frac{2}{1+e^{-s\frac{T_0}{2}}}\right)-1=\frac{1}{s}\left(\frac{1-e^{-s\frac{T_0}{2}}}{1+e^{-s\frac{T_0}{2}}}\right)$$

Theorem 6: Convolution

Given two signals , x_1(t) and x_2(t) , which are zero for $t < 0$

$$y(t) \triangleq x_1(t)*x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda)x_2(t-\lambda)d\lambda$$

Since $x_1(t) = 0$ for $t < 0$ $\Rightarrow x_1(\lambda) = 0$ for $\lambda < 0$

Since $x_2(t) = 0$ for $t < 0$ $\Rightarrow x_2(t-\lambda) = 0$ for $t-\lambda < 0$

$$\Rightarrow x_2(t-\lambda) = 0 \quad \text{for} \quad t < \lambda \quad \text{OR} \quad \lambda > t$$

$$\Rightarrow y(t) \triangleq x_1(t)*x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda)x_2(t-\lambda)d\lambda = \int_{0}^{t} x_1(\lambda)x_2(t-\lambda)d\lambda$$

Theorem $\mathrm{L}\left[x_1(t)*x_2(t)\right] = X_1(s)X_2(s)$

$$\mathrm{L}\left[x_1(t)*x_2(t)\right] = X_1(s)X_2(s)$$

$x_2(t-\lambda)=0 \quad \text{for} \quad \lambda > t$

Proof

$$\mathrm{L}\left[x_1(t)*x_2(t)\right] = \mathrm{L}\left[\int_0^{\boxed{t}} x_1(\lambda)x_2(t-\lambda)d\lambda\right] = \mathrm{L}\left[\int_0^{\boxed{\infty}} x_1(\lambda)x_2(t-\lambda)d\lambda\right]$$

$$= \int_0^\infty \left[\int_0^\infty x_1(\lambda)x_2(t-\lambda)d\lambda\right]e^{-st}dt = \int_0^\infty x_1(\lambda)\left[\int_0^\infty x_2(t-\lambda)d\lambda\right]e^{-st}dt$$

$$\text{Let} \quad \eta = t-\lambda \;\Rightarrow\; dt = d\eta \quad t = \eta+\lambda \;\Rightarrow\; e^{-st} = e^{-s(\eta+\lambda)} = e^{-s\eta}e^{-s\lambda}$$

$$\mathrm{L}\left[x_1(t)*x_2(t)\right] = \int_0^\infty x_1(\lambda)\left[\int_0^\infty x_2(t-\lambda)d\lambda\right]e^{-st}dt$$

$$= \int_0^\infty x_1(\lambda)\left[\int_0^\infty x_2(\eta)d\eta e^{-s\eta}\right]e^{-s\lambda}d\lambda$$

$$\mathrm{L}\left[x_1(t)*x_2(t)\right]=\int_0^\infty x_1(\lambda)\left[\int_0^\infty x_2(\eta)d\eta e^{-s\eta}\right]e^{-s\lambda}d\lambda$$

$$=\int_0^\infty x_1(\lambda)X_2(s)e^{-s\lambda}d\lambda$$

$$=X_2(s)\int_0^\infty x_1(\lambda)e^{-s\lambda}d\lambda \quad =X_2(s)X_1(s)$$

➡ $$\mathrm{L}\left[x_1(t)*x_2(t)\right]=X_1(s)X_2(s)$$

Region of Convergence $R_{x_1*x_2}=R_{x_1}\cap R_{x_2}$

Theorem 7: Product

$$\mathrm{L}\left[x_1(t)x_2(t)\right]=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X_1(s-z)X_2(z)dz \quad =\frac{1}{2\pi j}X_1(t)*X_2(t)$$

Proof Not shown

Theorem 8: Initial Value Theorem (i.e x(0))

(I) x(t) is continues at t = 0

$$\lim_{s \to \infty} sX(s) = x(0^-) = x(0^+)$$

Some times , we need x(0) ($x(0^-)$ or $x(0^+)$) , however what we have is X(s) . This theorem let you find x(0) (initial value) without finding x(t)

Proof

$$\mathrm{L}\left[\frac{dx(t)}{dt}\right] = \int_0^\infty \left[\frac{dx(t)}{dt}\right] e^{-st} dt = sX(s) - x(0^-)$$

$$\mathrm{L}\left[\frac{dx(t)}{dt}\right] = \int_{0}^{\infty}\left[\frac{dx(t)}{dt}\right]e^{-st}dt = sX(s) - x(0^{-})$$

Now taking the limit as $s \to \infty$ for both sides ,

$$\lim_{s\to\infty}\int_{0}^{\infty}\left[\frac{dx(t)}{dt}\right]e^{-st}dt = \lim_{s\to\infty}\left[sX(s) - x(0^{-})\right]$$

$$\int_{0}^{\infty}\lim_{s\to\infty}\left[\frac{dx(t)}{dt}\right]e^{-st}dt = = \lim_{s\to\infty} sX(s) - \lim_{s\to\infty} x(0^{-})$$

$$\int_{0}^{\infty}\left[\frac{dx(t)}{dt}\right]\left(\lim_{s\to\infty} e^{-st}\right)dt = \lim_{s\to\infty} sX(s) - x(0^{-})$$

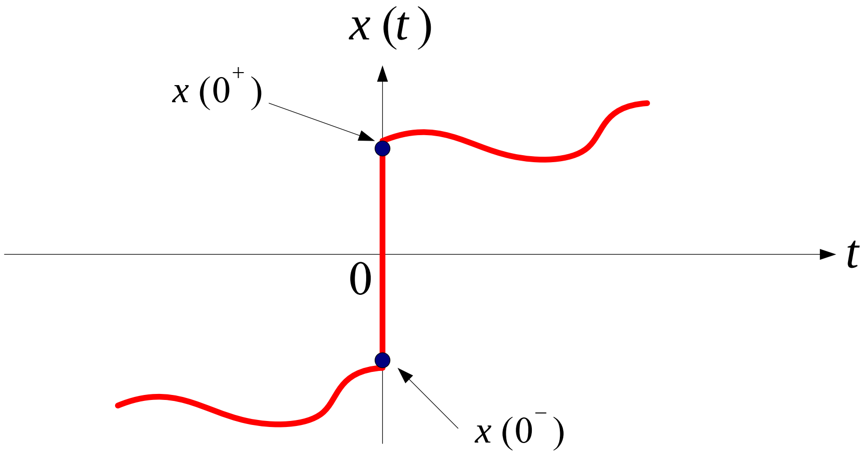
$$\int_0^\infty \left[\frac{dx(t)}{dt}\right]\left(\lim_{s\to\infty} e^{-st}\right)dt = \lim_{s\to\infty} sX(s) - x(0^-)$$

$$\int_0^\infty \left[\frac{dx(t)}{dt}\right](0)\,dt = \lim_{s\to\infty} sX(s) - x(0^-)$$

$$0 = \lim_{s\to\infty} sX(s) - x(0^-)$$

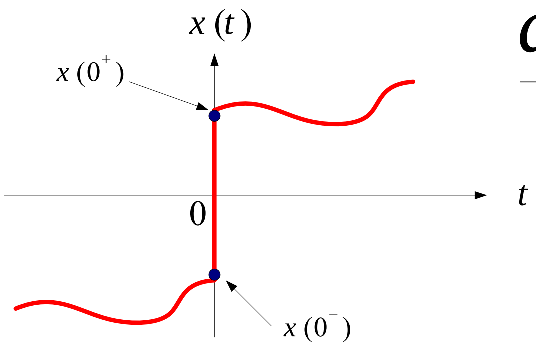
➡ $\lim_{s\to\infty} sX(s) = x(0^-) = x(0^+) = x(0)$ from continuity

(II) x(t) is discontinues at t = 0 ,



$\frac{dx(t)}{dt}$ containe an impulse $[x(0^+)-x(0^-)]\delta(t)$

then , $\lim_{s\to\infty} sX(s) = x(0^+)$



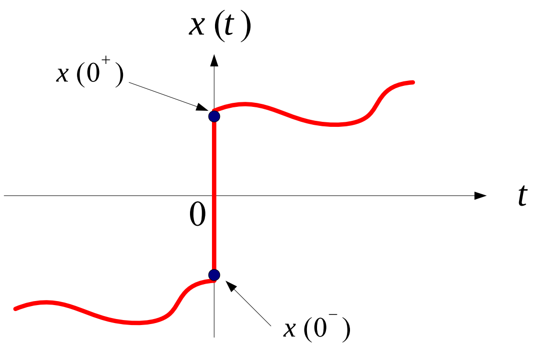
$$\frac{dx(t)}{dt} = x'(t)\Big|_{t \neq 0} + [x(0^+) - x(0^-)]\delta(t)$$

$$\int_0^\infty \left[\frac{dx(t)}{dt}\right] e^{-st} dt = sX(s) - x(0^-)$$

$$\lim_{s\to\infty} \int_0^\infty \left[\frac{dx(t)}{dt}\right] e^{-st} dt = \lim_{s\to\infty} sX(s) - x(0^-)$$

Left side $$\lim_{s\to\infty} \int_0^\infty \left[x'(t)\Big|_{t \neq 0} + [x(0^+) - x(0^-)]\delta(t)\right] e^{-st} dt$$

$$\int_0^\infty \lim_{s\to\infty} x'(t)\Big|_{t \neq 0} e^{-st} dt + \lim_{s\to\infty} [x(0^+) - x(0^-)] \int_0^\infty \delta(t) e^{-st} dt$$



$$\frac{dx(t)}{dt} = x'(t)\Big|_{t\neq 0} + [x(0^+)-x(0^-)]\delta(t)$$

Left side

$$\int_0^\infty x'(t)\Big|_{t\neq 0}(\lim_{s\to\infty} e^{-st})dt + \lim_{s\to\infty}[x(0^+)-x(0^-)]\int_0^\infty \delta(t)e^{-st}dt$$

$$\int_0^\infty x'(t)\Big|_{t\neq 0}(0)dt + \lim_{s\to\infty}[x(0^+)-x(0^-)](1) = x(0^+)-x(0^-)$$

$$x(0^+)-\cancel{x(0^-)} = \lim_{s\to\infty} sX(s) - \cancel{x(0^-)}$$

$$\Rightarrow \quad \lim_{s\to\infty} sX(s) = x(0^+)$$

Example 5-6

Theorem 9: Final Value Theorem

If $x(t)$ and $\frac{dx(t)}{dt}$ are Laplace Transformble

then,

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Provided that $\lim_{t\to\infty} x(t)$ exists or $s\mathrm{X}(s)$ has no poles on the $\mathrm{j}\omega$ axis or in the Right Half Plane

Proof: Not shown

Example 5-7

Theorem 10: Scaling

If $x(t) \Leftrightarrow X(s)$

Then $x(at) \quad a > 0 \Leftrightarrow \frac{1}{a} X\left(\frac{s}{a}\right)$

Note $a > 0$ because , if $a < 0$, then $x(at)$ will be reflected on the negative Part which Laplace Transform ignore

Example

$$\text{Let } \ x(t)=e^{-t}u(t) \quad \Rightarrow \quad \mathrm{L}[x(t)] = \frac{1}{(1+s)}$$

$$\text{Now } \ x(3t)=e^{-3t}u(3t)=e^{-3t}u(t)$$

$$\mathrm{L}[\,e^{-3t}u(t)\,] = \int_0^\infty e^{-3t}e^{-st}dt = \int_0^\infty e^{-(3+s)t}dt = \left.\frac{e^{-(3+s)t}}{-(3+s)}\right|_0^\infty = \frac{1}{(3+s)}$$

$$x(3t) \Leftrightarrow \frac{1}{3}X\left(\frac{s}{3}\right) = \frac{1}{3}\frac{1}{\left(1+\frac{s}{3}\right)} = \frac{1}{(1+s)}$$

5.4 Inversion of Rational Function (Inverse Laplace Transform)

Let $Y(s)$ be Laplace Transform of some function $y(\mathrm{t})$.

We want to find $y(\mathrm{t})$ without using the inversion formula .

⇒ We want to find $y(\mathrm{t})$ using the Laplace Transform known table and properties

Objective : Put $Y(s)$ in a form or a sum of forms that we know it is in the Laplace Transform Table

$Y(s)$ in general is a ratio of two polynomials ⇒ Rational Function

Example $$Y(s) = \frac{s^2 - 2s}{2s^3 - 5s^2 + 3s + 2}$$

When the degree of the numerator of rational function is less the Degree of the dominator

⇒ *Proper Rational Function*

Highest Degree is **2**

$$\text{Example}\quad Y(s) = \frac{s^2 - 2s}{2s^3 - 5s^2 + 3s + 2}$$

Highest Degree is **3**

Examples of proper rational Functions

$$Y_1(s) = \frac{1}{s+1}$$

$$Y_2(s) = \frac{2s^2+6s+6}{(s+2)(s^2+2s^2+2)}$$

Examples of not proper rational Functions

$$Y_3(s) = \frac{s+2}{s+1}$$

However we can obtain a proper rational Function through long division

$$Y_3(s) = \frac{s+2}{s+1} = 1 + \frac{1}{s+1}$$

We will discuss different techniques of factoring $Y(s)$ into simple known forms

Example 5-9 Simple Factors

$$\text{Let } Y(s) = \frac{10}{(s^2+2s^2+2)}$$

If we check the Table , we see there is no form similar to $Y(s)$

However if we expand $Y(s)$ in partial fractions:

$$\frac{10}{(s^2+2s^2+2)} = \frac{A}{(s+2)} + \frac{B}{(s+8)}$$

$\frac{A}{(s+2)}$ and $\frac{B}{(s+8)}$ Are available on the Table

Next we develop Techniques of finding A and B

Techniques for Partial Fraction Expansion

(1) Common Denominator

$$Y(s)=\frac{10}{(s^2+10s+16)}=\frac{10}{(s+2)(s+8)}=\frac{A}{(s+2)}+\frac{B}{(s+8)}$$

$$\Rightarrow \quad Y(s)=\frac{10}{(s^2+10s+16)}=\frac{A(s+8)+B(s+2)}{(s+2)(s+8)}$$

$$\Rightarrow \quad 10=A(s+8)+B(s+2)=(A+B)s+(8A+2B)$$

$$\Rightarrow \quad \left.\begin{aligned} A+B&=0 \\ 8A+2B&=10 \end{aligned}\right\} \xrightarrow{\textbf{Solve}} A=\frac{5}{3} \text{ and } B=-\frac{5}{3}$$

$$Y(s) = \frac{10}{(s^2 + 10s + 16)} = \frac{5/3}{(s+2)} - \frac{5/3}{(s+8)}$$

$$\Rightarrow \quad y(\mathrm{t}) = \frac{5}{3}e^{-2t}u(t) - \frac{5}{3}e^{-8t}u(t) = \frac{5}{3}\left(e^{-2t} - e^{-8t}\right)u(t)$$

(2) Substituting Specific values of s

$$Y(s) = \frac{10}{(s^2+10s+16)} = \frac{10}{(s+2)(s+8)} = \frac{A}{(s+2)} + \frac{B}{(s+8)}$$

Solve

$$s = 0 \Rightarrow \frac{10}{(2)(8)} = \frac{A}{2} + \frac{B}{8} \Rightarrow 4A+B=5$$

$$s = 2 \Rightarrow \frac{10}{(4)(10)} = \frac{A}{4} + \frac{B}{10} \Rightarrow 5A+2B=5$$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \quad A = \frac{5}{3} \qquad B = -\frac{5}{3}$$

(3) Heavisdie’s Expansion Theorem

$$Y(s) = \frac{10}{(s^2+10s+16)} = \frac{10}{(s+2)(s+8)} = \frac{A}{(s+2)} + \frac{B}{(s+8)}$$

Multiply both side by $(s+2)$ and set $s = -2$

$$\frac{10}{\cancel{(s+2)}(s+8)}\mathrm{X}\cancel{(s+2)}\Big|_{s=-2} = \frac{A}{\cancel{(s+2)}}\mathrm{X}\cancel{(s+2)}\Big|_{s=-2} + \frac{B}{(s+8)}\mathrm{X}(s+2)\Big|_{s=-2}$$

➡ $$\frac{10}{(-2+8)} = A + \frac{B(-2+2)}{(-2+8)}$$

➡ $$\frac{10}{(6)} = A + 0$$ ➡ $$A = \frac{5}{3}$$

$$Y(s) = \frac{10}{(s^2+10s+16)} = \frac{10}{(s+2)(s+8)} = \frac{A}{(s+2)} + \frac{B}{(s+8)}$$

Multiply both side by $(s+8)$ and set $s = -8$

$$\frac{10}{(s+2)\cancel{(s+8)}}\mathrm{X}\cancel{(s+8)}\Big|_{s\ =-8} = \frac{A}{(s+2)}\mathrm{X}(s+8)\Big|_{s\ =-8} + \frac{B}{\cancel{(s+8)}}\mathrm{X}\cancel{(s+8)}\Big|_{s\ =-8}$$

$$\Rightarrow \qquad \frac{10}{(-8+2)} = 0+B$$

$$\Rightarrow \qquad \frac{10}{(-6)} = B \qquad \Rightarrow \qquad B = -\frac{5}{3}$$

Example 5-10 (Imaginary Roots)

Let $$Y(s)=\frac{(15s^2+25s+20)}{(s^2+1)(s^2+10s+16)}=\frac{(15s^2+25s+20)}{(s+j)(s-j)(s+2)(s+8)}$$

$$=\frac{A_1}{(s+j)}+\frac{A_2}{(s-j)}+\frac{A_3}{(s+2)}+\frac{A_4}{(s+8)}$$

Using Heavisdie's Expansions, by multiplying the left hand side and Right hand side by the factors

$$(s+j)\ ,\ (s-j)\ ,\ (s+2)\ ,\ (s+8)$$

and substitute $s=-j\ ,\ s=j\ ,\ s=-2\ ,\ s=-8$ respectively

We obtain $A_1=\frac{1}{2}(1+j)\ ,\ A_2=\frac{1}{2}(1-j)\ ,\ A_3=1\ ,\ A_4=-2$

From Table

$e^{-2t}u(t)$ $\quad$ $e^{-8t}u(t)$

$$Y(s)=\frac{(1/2)(1+j)}{(s+j)}+\frac{(1/2)(1-j)}{(s-j)}+\frac{1}{(s+2)}-\frac{2}{(s+8)}$$

$\frac{(1/2)(1+j)}{(s+j)}$ and $\frac{(1/2)(1-j)}{(s-j)}$ **Can be inverted in two methods:**

(a)

$$\frac{(1/2)(1+j)}{(s+j)} \Rightarrow (1/2)(1+j)\,e^{-jt}u(t)$$

$$\frac{(1/2)(1-j)}{(s-j)} \Rightarrow (1/2)(1-j)\,e^{jt}u(t)$$

combine

$$\left.\begin{aligned}\frac{(1/2)(1+j)}{(s+j)} &\Rightarrow (1/2)(1+j)\,e^{-jt}u(t)\\ \frac{(1/2)(1-j)}{(s-j)} &\Rightarrow (1/2)(1-j)\,e^{jt}u(t)\end{aligned}\right\}$$

$$\frac{(1/2)(1+j)}{(s+j)}+\frac{(1/2)(1-j)}{(s-j)} \Rightarrow \frac{1}{2}(1+j)\,e^{-jt}u(t)+\frac{1}{2}(1-j)\,e^{jt}u(t)$$

$$=\frac{1}{2}(e^{jt}+e^{-jt})\,u(t)+\frac{j}{2}(e^{-jt}-e^{jt})u(t)$$

$$=\frac{1}{2}(e^{jt}+e^{-jt})\,u(t)+\frac{1}{2j}(e^{jt}-e^{-jt})u(t)$$

$$=\cos(t)\,u(t)+\sin(t)\,u(t)$$

(b) $\dfrac{(1/2)(1+j)}{(s+j)}$ and $\dfrac{(1/2)(1-j)}{(s-j)}$ **Can be combined as**

$$\frac{(1/2)(1+j)}{(s+j)} + \frac{(1/2)(1-j)}{(s-j)}$$

$$= \frac{(1/2)(1+j)(s-j)+(1/2)(1-j)(s+j)}{(s+j)(s-j)}$$

$$= \frac{s+1}{s^2+1} = \frac{s}{s^2+1} + \frac{1}{s^2+1} \quad \Rightarrow \cos(t)\,u(t)+\sin(t)\,u(t)$$

$$Y(s)=\overbrace{\frac{(1/2)(1+j)}{(s+j)}+\frac{(1/2)(1-j)}{(s-j)}}^{\cos(t)\,u(t)+\sin(t)\,u(t)}+\overset{e^{-2t}u(t)}{\frac{1}{(s+2)}}-\overset{e^{-8t}u(t)}{\frac{2}{(s+8)}}$$

$$y(t)=\cos(t)\,u(t)+\sin(t)\,u(t)+e^{-2t}u(t)+e^{-8t}u(t)$$

$$y(t)=\left(\cos(t)\;+\sin(t)\;+e^{-2t}+e^{-8t}\right)u(t)$$

Repeated Linear Factor

If $Y(s) = \dfrac{P(s)}{(s+\alpha)^n Q(s)}$ $\left\{ \text{example } Y(s) = \dfrac{10s}{(s+2)^2(s+8)} \right\}$

Then its partial fraction

$$Y(s) = \frac{A_1}{(s+\alpha)} + \frac{A_2}{(s+\alpha)^2} + \frac{A_2}{(s+\alpha)^3} + \cdots + \frac{A_n}{(s+\alpha)^n} + \frac{R(s)}{Q(s)}$$

Were

$$A_m = \frac{1}{(n-m)!} \frac{d^{(n-m)}}{ds^{(n-m)}} \left[(s+\alpha)^n Y(s) \right]_{s=-\alpha}$$

Example 5-11 Repeated Linear Factor

Let $$Y(s)=\frac{10s}{(s+2)^2(s+8)}$$

Then $$Y(s)=\frac{A_1}{(s+2)}+\frac{A_2}{(s+2)^2}+\frac{B}{(s+8)}$$

$$A_1=\frac{1}{(2-1)!}\,\frac{d^{(2-1)}}{ds^{(2-1)}}\left[\cancel{(s+2)^{2}}\,\frac{10s}{\cancel{(s+2)^{2}}(s+8)}\right]_{s=-2}=\frac{d}{ds}\left[\frac{10s}{(s+8)}\right]_{s=-2}$$

$$=\left[\frac{10(s+8)-10s}{(s+8)^2}\right]_{s=-2}=\left[\frac{10(-2+8)-10(-2)}{(-2+8)^2}\right]=\left[\frac{80}{36}\right]=\frac{20}{9}$$

$$Y(s)=\frac{\overset{\nearrow\boxed{\frac{20}{9}}}{A_1}}{(s+2)}+\frac{\overset{\nearrow\boxed{-\frac{10}{3}}}{A_2}}{(s+2)^2}+\frac{B}{(s+8)}$$

$$A_2=\frac{1}{(2-2)!}\,\frac{d^{(2-2)}}{ds^{(2-2)}}\left[\cancel{(s+2)^2}\,\frac{10s}{\cancel{(s+2)^2}(s+8)}\right]_{s=-2}=\left[\frac{10s}{(s+8)}\right]_{s=-2}$$

$$=\left[\frac{10(-2)}{(-2+8)}\right]\quad=\left[\frac{-20}{6}\right]\quad\boxed{=-\frac{10}{3}}$$

$$Y(s)=\frac{\overset{\boxed{\frac{20}{9}}}{A_1}}{(s+2)}+\frac{\overset{\boxed{-\frac{10}{3}}}{A_2}}{(s+2)^2}+\frac{\overset{\boxed{-\frac{20}{9}}}{B}}{(s+8)}$$

To find B, we use Heaviside

$$\cancel{(s+8)}\frac{10s}{(s+2)^2\cancel{(s+8)}}\Bigg|_{s=-8}=\overset{\boxed{0}}{\frac{A_1}{(s+2)}\text{X}\cancel{(s+8)}}\Bigg|_{s=-8}+\overset{\boxed{0}}{\frac{A_2}{(s+2)^2}\text{X}\cancel{(s+8)}}\Bigg|_{s=-8}+\frac{B}{\cancel{(s+8)}}\text{X}\cancel{(s+8)}\Bigg|_{s=-8}$$

$$\Rightarrow\quad B=-\frac{20}{9}\qquad\qquad \Rightarrow Y(s)=\frac{(20/9)}{(s+2)}-\frac{(10/3)}{(s+2)^2}-\frac{(20/9)}{(s+8)}$$

$$Y(s)=\frac{20}{9}\left[-\frac{1}{(s+8)}+\frac{1}{(s+2)}-\frac{(3/2)}{(s+2)^2}\right]$$

$$y(t)=\frac{20}{9}\left(-e^{-8t}+e^{-2t}-\frac{3}{2}te^{-2t}\right)u(t)$$

Example 5-12 Repeated Linear Factor

Let $$Y(s) = \frac{10s}{(s+2)^3(s+8)}$$

Can be found using Heaviside's expansion

Then $$Y(s) = \frac{A_1}{(s+2)} + \frac{A_2}{(s+2)^2} + \frac{A_3}{(s+2)^3} + \frac{B}{(s+8)}$$

$$B = (s+8)\,Y(s)\Big|_{s=-8} = \frac{10}{27}$$

$$A_3 = (s+2)^3\,Y(s)\Big|_{s=-2} = -\frac{10}{3}$$

$$A_2 = \frac{1}{(3-2)!}\frac{d^{(3-2)}}{ds^{(3-2)}}\left[(s+2)^3 Y(s)\right]_{s=-2} = \frac{d}{ds}\left[\frac{10s}{(s+8)}\right]_{s=-2} = \frac{20}{9}$$

$$Y(s)=\frac{\overset{-\frac{20}{27}}{\nearrow}A_1}{(s+2)}+\frac{\overset{\frac{20}{9}}{\nearrow}A_2}{(s+2)^2}+\frac{\overset{-\frac{10}{3}}{\nearrow}A_3}{(s+2)^3}+\frac{\overset{\frac{10}{27}}{\nearrow}B}{(s+8)}$$

A_1 Can be found using Heaviside differentiation techniques

$$A_1=\frac{1}{(3-1)!}\frac{d^{(3-1)}}{ds^{(3-1)}}\left[(s+2)^3Y(s)\right]_{s=-2}=\frac{1}{2}\frac{d^2}{ds^2}\left[\frac{10s}{(s+8)}\right]_{s=-2}$$

$$=\frac{1}{2}\left[\frac{-160}{(s+8)^3}\right]_{s=-2}\qquad=-\frac{20}{27}$$

$$Y(s)=\frac{10}{27}\left[\frac{1}{(s+8)}-\frac{1}{(s+2)}\right]+\frac{20}{9}\frac{1}{(s+2)^2}-\frac{10}{3}\frac{1}{(s+2)^3}$$

$$\Rightarrow y(t)=\left[\frac{10}{27}\left(e^{-8t}-e^{-2t}\right)-\frac{5}{3}t\left(t-\frac{4}{3}\right)e^{-2t}\right]u(t)$$

Example 5-13 Complex Conjugate Factors

$$Y(s)=\frac{2s^2+6s+6}{(s+2)(s^2+2s+2)}=\frac{2s^2+6s+6}{(s+2)\left[(s+1)^2+1\right]}$$

$$=\frac{2s^2+6s+6}{(s+2)(s+1+j)(s+1-j)}$$

Using Heaviside Expansion $Y(s)=\frac{A_1}{(s+2)}+\frac{A_2}{(s+1+j)}+\frac{A_3}{(s+1-j)}$

We can find A_1, A_2, A_3

However it is easier to keep both of the complex-conjugate factors together

$$Y(s)=\frac{A}{s+2}+\frac{Bs+C}{s^2+2s+2}$$

$$Y(s)=\frac{A_1}{(s+2)}+\frac{A_2}{(s+1+j)}+\frac{A_3}{(s+1-j)}$$

However it is easier to keep both of the complex-conjugate factors together

$$Y(s)=\frac{A}{s+2}+\frac{Bs+C}{s^2+2s+2}$$

This allows the inverse Laplace Transform to be found easily with the help of pairs

$$\frac{t^n e^{-\alpha t}u(t)}{n!} \Leftrightarrow \frac{1}{(s+\alpha)^{n+1}}$$

$$e^{-\alpha t}\cos(\omega_0 t)u(t) \Leftrightarrow \frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$$

$$e^{-\alpha t}\sin(\omega_0 t)u(t) \Leftrightarrow \frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$$

$$Y(s)=\frac{\overset{1}{A}}{s+2}+\frac{\overset{2}{Bs+C}}{s^{2}+2s+2}$$

A Can be found using Heaviside

$$A=(s+2)\,Y(s)\Big|_{s=-2}=\left.\frac{2s^{2}+6s+6}{s^{2}+2s+2}\right|_{s=-2}=1$$

B and C can be found using substitution of s or the common denominator

$$s=0 \qquad \Rightarrow Y(0)=\frac{A}{0+2}+\frac{B(0)+C}{0^{2}+2(0)+2}=\frac{A}{2}+\frac{C}{2}$$

$$\Rightarrow C=2\,Y(0)-A=\frac{2(6)}{(4)}-1=2$$

$$Y(s) = \frac{2s^2+6s+6}{(s+2)(s^2+2s+2)} = \frac{1}{s+2} + \frac{\overset{1}{B}s+2}{s^2+2s+2}$$

To find B, we multiply both sides of $Y(s)$ by s and let s → ∞

$$\lim_{s\to\infty} s\frac{2s^2+6s+6}{(s+2)(s^2+2s+2)} = \lim_{s\to\infty}\left[\frac{s}{s+2} + \frac{Bs^2+2s}{s^2+2s+2}\right]$$

$$\lim_{s\to\infty}\frac{2s^3+6s^2+6s}{(s^3+4s^2+6s+4)} = \lim_{s\to\infty}\frac{s}{s+2} + \lim_{s\to\infty}\frac{Bs^2+2s}{s^2+2s+2}$$

$$\lim_{s\to\infty}\frac{2+6(1/s)+6(1/s^2)}{(1+4(1/s)+6(1/s^2)+(4/s^3))} = \lim_{s\to\infty}\frac{1}{1+(2/s)} + \lim_{s\to\infty}\frac{B+2(1/s)}{1+(2/s)+(2/s^2)}$$

$$\frac{2+6(0)+6(0)}{(1+4(0)+6(0)+(0))} = \frac{1}{1+(0)} + \frac{B+2(0)}{1+(0)+(0)}$$

$$\Rightarrow \frac{2}{1} = \frac{1}{1} + \frac{B}{1} \Rightarrow B = 1$$

$$Y(s) = \frac{2s^2+6s+6}{(s+2)(s^2+2s+2)} = \frac{1}{s+2} + \frac{Bs+2}{s^2+2s+2}$$

We also can find B by selecting any value of s

$$Y(1) = \frac{2(1)^2+6(1)+6}{((1)+2)((1)^2+2(1)+2)} = \frac{1}{(1)+2} + \frac{B(1)+2}{(1)^2+2(1)+2}$$

➡ $$\frac{14}{15} = \frac{1}{3} + \frac{B+2}{5}$$

➡ $$14 = 5 + 3B + 6$$

$$B = 1$$

$$Y(s) = \frac{2s^2+6s+6}{(s+2)(s^2+2s+2)} = \frac{1}{s+2} + \frac{s+2}{s^2+2s+2}$$

$$= \frac{1}{s+2} + \frac{s+2}{(s+1)^2+1} \quad = \frac{1}{s+2} + \frac{(s+1)+1}{(s+1)^2+1}$$

$$= \frac{1}{s+2} + \frac{(s+1)}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$$

$$\Rightarrow y(t) = e^{-2t}u(t) + e^{-t}\cos(t)u(t) + e^{-t}\sin(t)u(t)$$

$$\Rightarrow y(t) = \left[e^{-2t} + e^{-t}\cos(t) + e^{-t}\sin(t)\right]u(t)$$