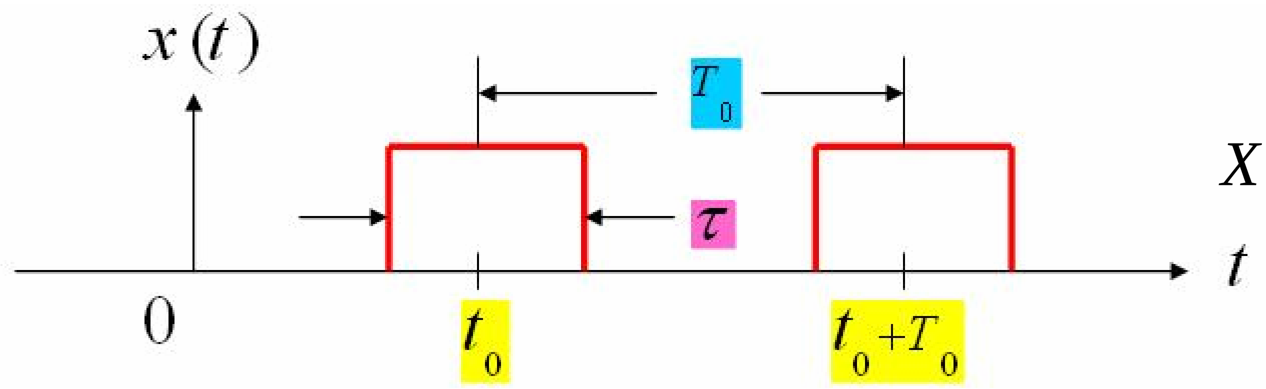
EE 207 Dr. Adil Balghonaim

Chapter 4 The Fourier Transform

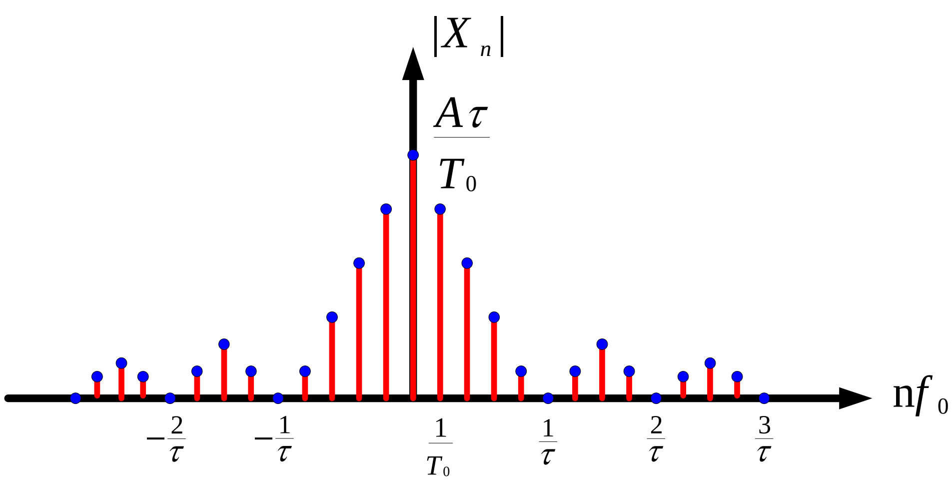
From Example 3-4

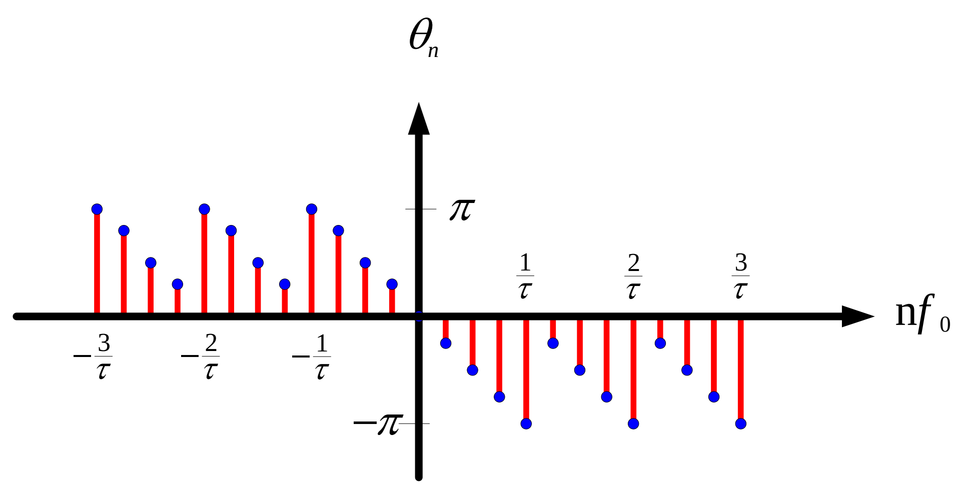


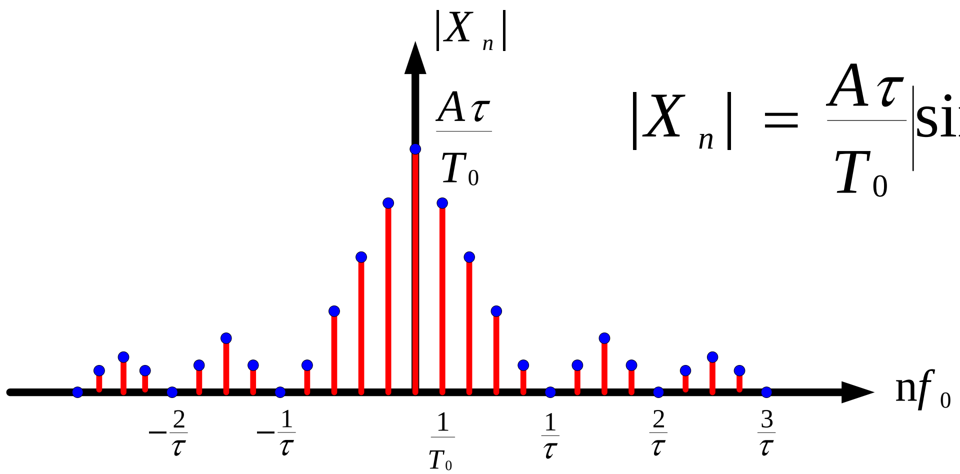
$$X_n = \frac{A\tau}{T_0}\operatorname{sinc}\left(nf_0\tau\right)e^{-j2\pi nf_0t_0}$$

$$|X_n| = \frac{A\tau}{T_0}\left|\operatorname{sinc}\left(nf_0\tau\right)\right|$$

$$\theta_n = \begin{cases} -\pi nf_0\tau & \text{if } \operatorname{sinc}\left(nf_0\tau\right) > 0 \\ -\pi nf_0\tau + \pi & \text{if } nf_0 > 0 \text{ and } \operatorname{sinc}\left(nf_0\tau\right) < 0 \\ -\pi nf_0\tau - \pi & \text{if } nf_0 < 0 \text{ and } \operatorname{sinc}\left(nf_0\tau\right) < 0 \end{cases}$$

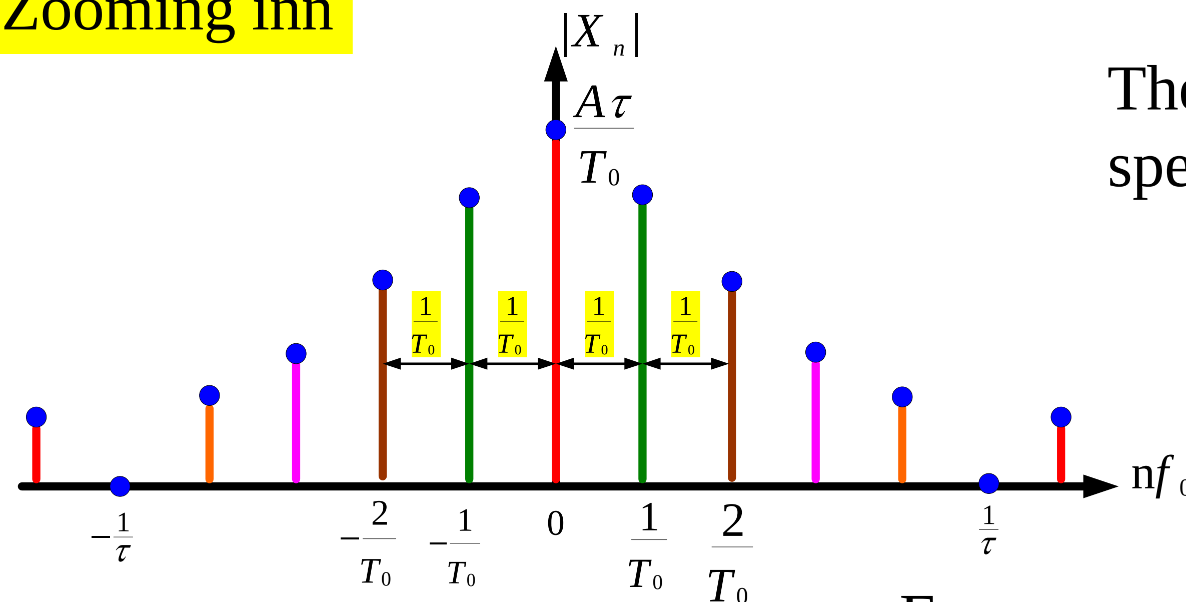






$$|X_n| = \frac{A\tau}{T_0}\left|\mathrm{sinc}\left(nf_0\tau\right)\right|$$

Zooming inn



The spacing between spectrum lines is $\frac{1}{T_0} = f_0$

Frequency increment $\Delta(nf_0) = f_0$

In the variable nf_0

Let $x_p(t)$ be a periodical wave, then expanding the periodical function

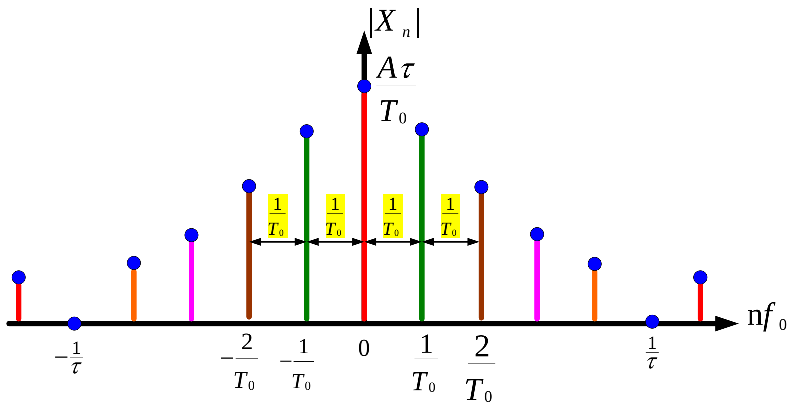
$$x_p(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \qquad X_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

$$\omega_0 = 2\pi f_0 \quad \Rightarrow \quad x_p(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t}$$

Rewriting $x_p(t)$ and X_n

$$x_p(t) = \sum_{n=-\infty}^{\infty} \frac{X_n}{f_0} e^{j2\pi n f_0 t} \Delta(nf_0) = \sum_{nf_0=-\infty}^{\infty} \frac{X_n}{f_0} e^{j2\pi n f_0 t} \Delta(nf_0)$$

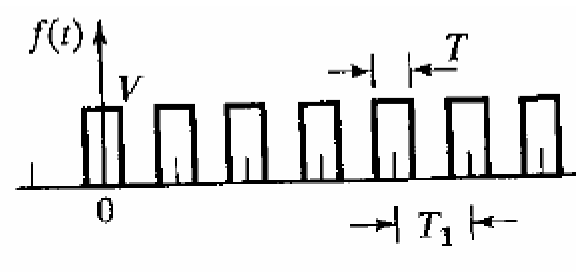
$$\tilde{X}(nf_0) \triangleq \frac{X_n}{f_0} = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi n f_0 t} dt$$

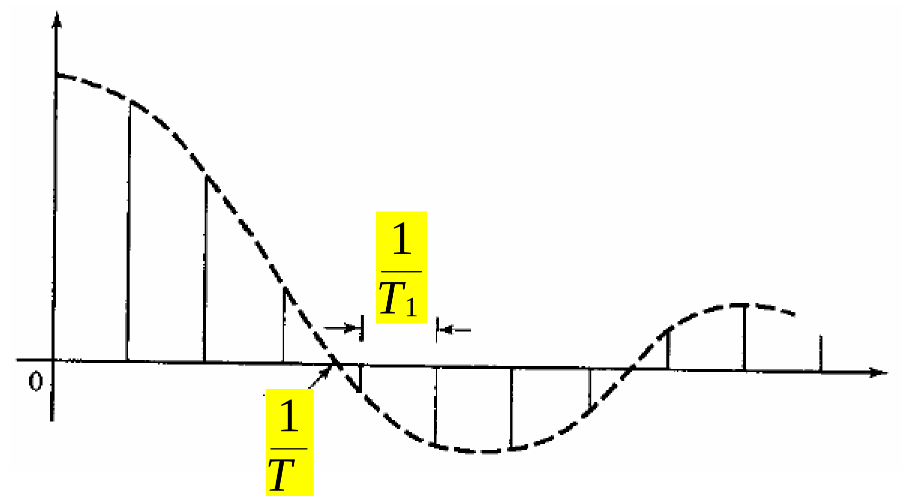


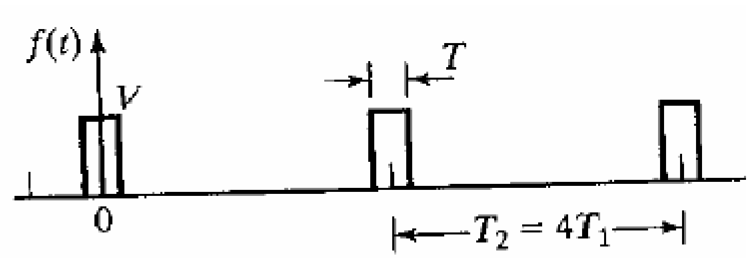
$$x_p(t) = \sum_{nf_0=-\infty}^{\infty} \frac{X_n}{f_0} e^{j2\pi nf_0 t} \Delta(nf_0)$$

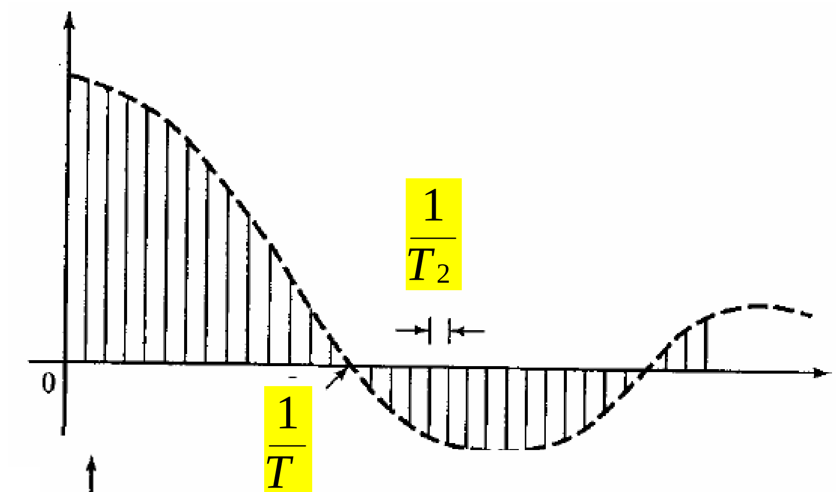
$$\Delta(nf_0) = f_0$$

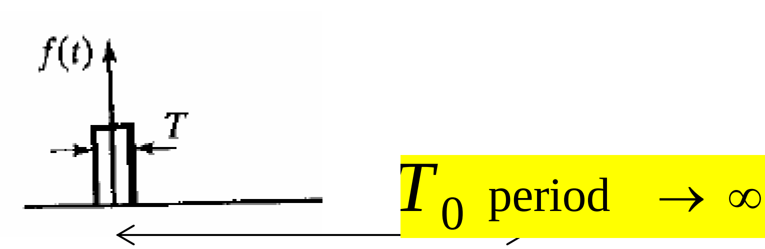
$$\tilde{X}(nf_0) \triangleq \frac{X_n}{f_0} = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi nf_0 t} dt$$

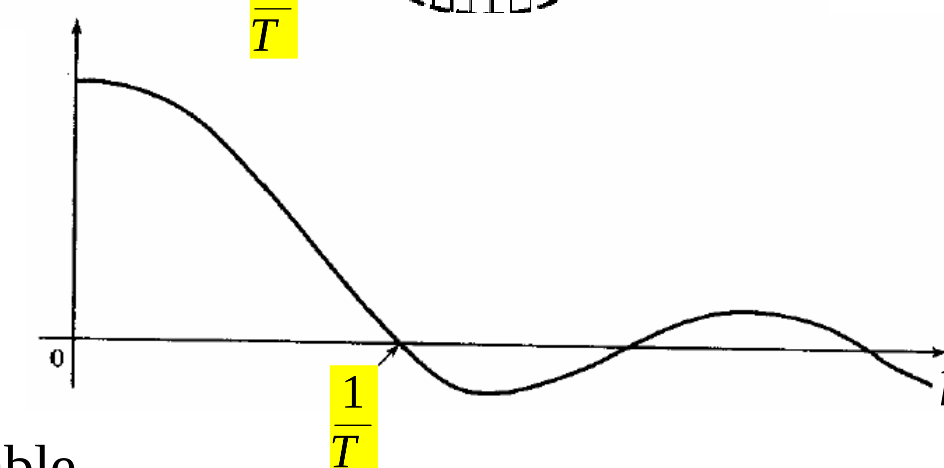










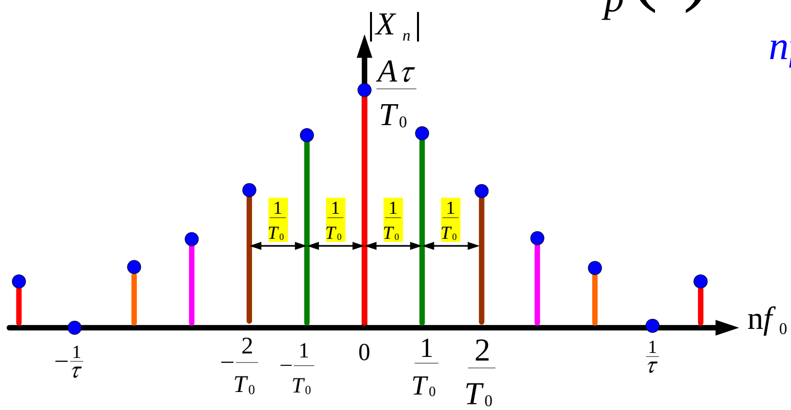


$T_0 \to \infty \quad \Rightarrow f_0 \to 0$

$n \to \infty$

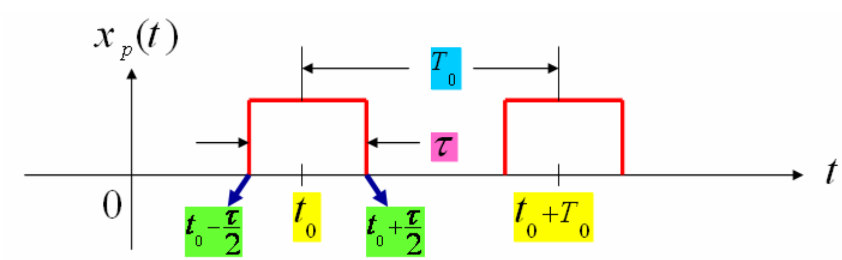
$\Rightarrow \quad nf_0 \to f$ Continuous Variable

$$x_p(t) = \sum_{nf_0=-\infty}^{\infty} \frac{X_n}{f_0} e^{j2\pi n f_0 t} \Delta(nf_0) \qquad \Delta(nf_0) = f_0$$



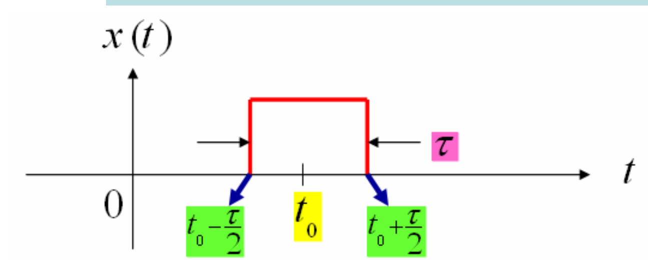
$$\tilde{X}(nf_0) \triangleq \frac{X_n}{f_0} = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi n f_0 t} dt$$

$x_p(t)$ **Periodic Power Signal**



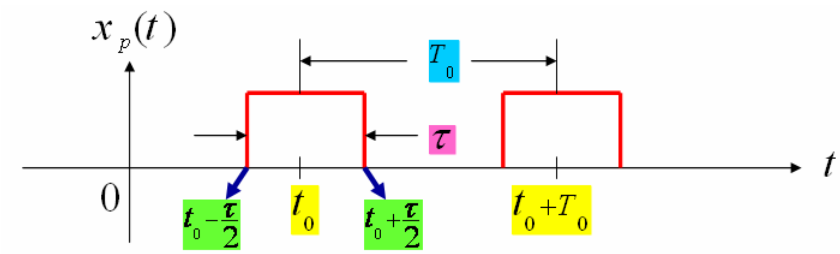
$$T_0 \to \infty$$

$x(t)$ **Aperiodic Energy Signal**



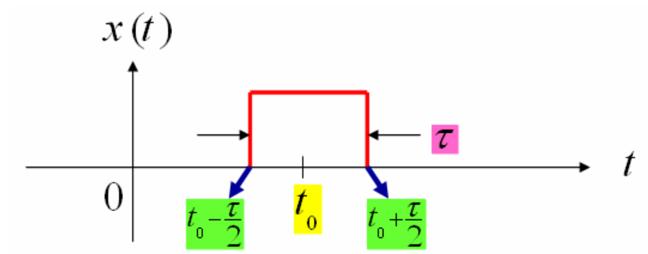
$$\Rightarrow f_0 \to 0 \quad nf_0 \to f \quad \Delta(nf_0) \to df \quad \sum \to \int$$

$x_p(t)$ **Periodic Power Signal**



$T_0 \to \infty$

$x(t)$ **Aperiodic Energy Signal**



$$x_p(t) = \sum_{nf_0=-\infty}^{\infty} \tilde{X}(nf_0) e^{j2\pi n f_0 t} \Delta(nf_0) \qquad \tilde{X}(nf_0) = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi n f_0 t} dt$$

$$\Rightarrow f_0 \to 0 \quad nf_0 \to f \quad \Delta(nf_0) \to df \quad \sum \to \int$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \qquad X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

Fourier Transform Pairs

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \qquad X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

$$X(f) = |X(f)| e^{j\theta(f)}$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} x(t) [\cos 2\pi ft - j \sin 2\pi f] dt$$

$$= \underbrace{\int_{-\infty}^{\infty} x(t) \cos 2\pi ft \; dt}_{A(f)} - j \underbrace{\int_{-\infty}^{\infty} x(t) \sin 2\pi f \; dt}_{B(f)} = A(f) - jB(\mathrm{f})$$

$$X(-f) = \int_{-\infty}^{\infty} x(t) \cos 2\pi ft \; dt + j \int_{-\infty}^{\infty} x(t) \sin 2\pi f \; dt$$

$$= A(f) + j \; B(\mathrm{f})$$

$$X(f) = |X(f)| e^{j\theta(f)}$$

$$X(f) = A(f) - jB(f)$$

$$X(-f) = A(f) + j\,B(f)$$

$$|X(f)| = \sqrt{A^2(f) + B^2(f)} = |X(-f)|$$

$\Rightarrow$ $|X(f)|$ Even function

$$\theta(f) = \tan^{-1}\frac{-B(f)}{A(f)} \qquad \theta(-f) = \tan^{-1}\frac{B(f)}{A(f)} = -\theta(f)$$

$\Rightarrow$ $\theta(f)$ odd function

$$X(f) = \underbrace{\int_{-\infty}^{\infty} x(t)\cos 2\pi ft \; dt}_{A(f)} - j\underbrace{\int_{-\infty}^{\infty} x(t)\sin 2\pi f \; dt}_{B(f)} = A(f) - jB(\mathrm{f})$$

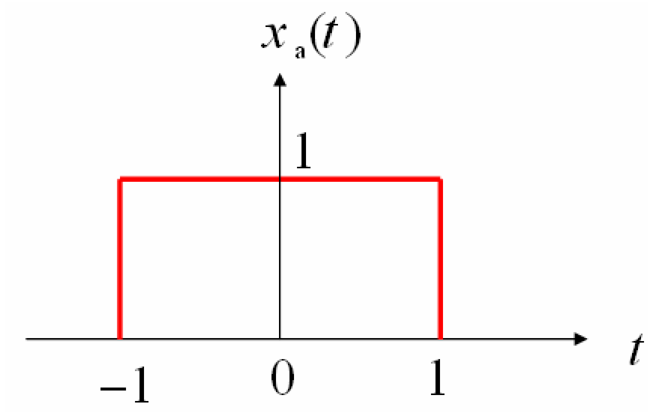
If $x(t)$ is an even function $B(f) = 0$

$$X(f) = A(f) = \int_{-\infty}^{\infty} x(t)\cos 2\pi ft \; dt$$ is real

If $x(t)$ is an odd function $A(f) = 0$

$$X(f) = -jB(f) = -j\int_{-\infty}^{\infty} x(t)\sin 2\pi ft \; dt$$ is an imaginary

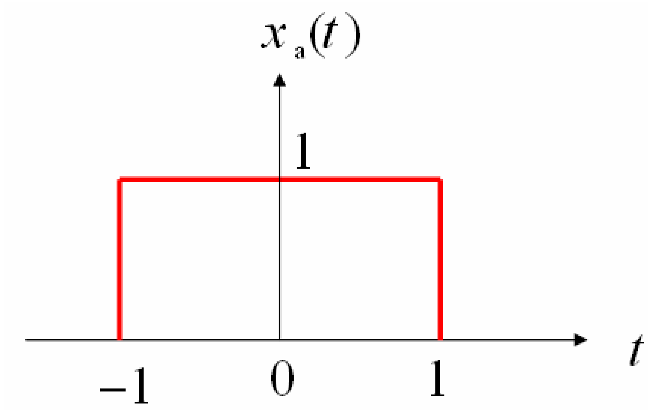
Example 4-1 Find the Fourier Transform for the following function



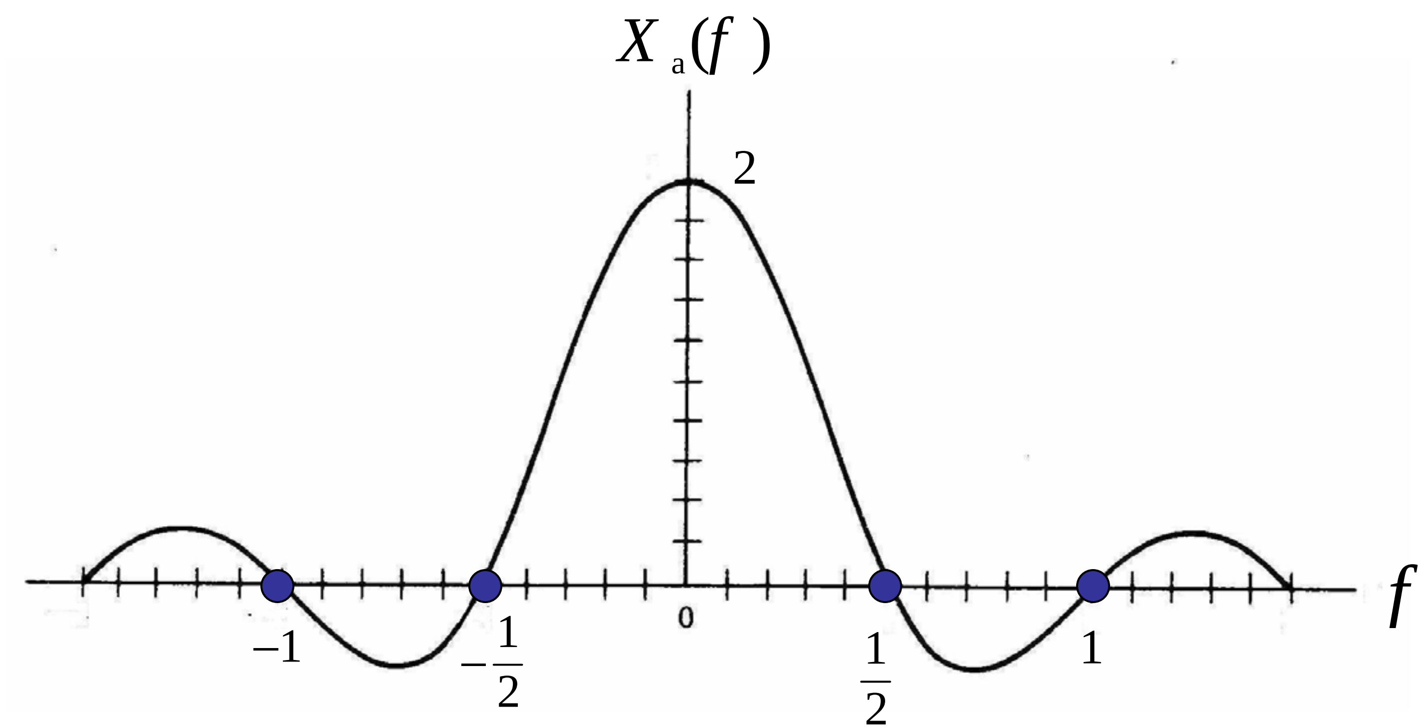
$$X_a(f)=\int_{-\infty}^{\infty} x_a(t)e^{-j2\pi ft}dt = \int_{-1}^{1} e^{-j2\pi ft}dt = \left.\frac{e^{-j2\pi ft}}{-j2\pi f}\right|_{-1}^{1}$$

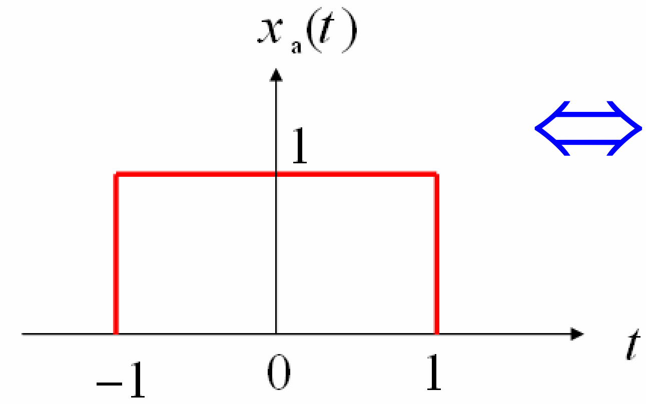
$$=\frac{e^{-j2\pi f(1)}-e^{-j2\pi f(-1)}}{-j2\pi f} = \frac{1}{\pi f}\frac{e^{j2\pi f}-e^{-j2\pi f}}{j2} = \frac{1}{\pi f}\sin(2\pi f)$$

$$=2\frac{\sin(2\pi f)}{2\pi f} = 2\,\mathrm{sinc}(2f)$$

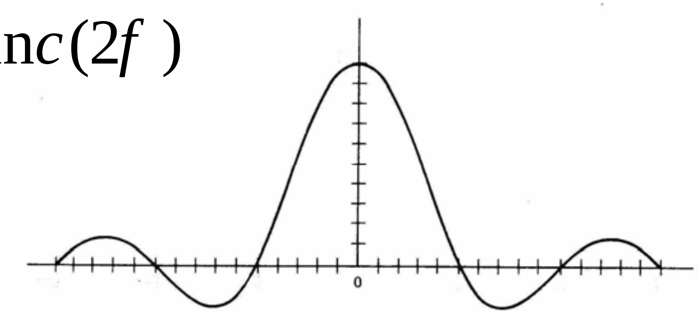


$$\Leftrightarrow \quad X_a(f) = 2\mathrm{sinc}(2f)$$

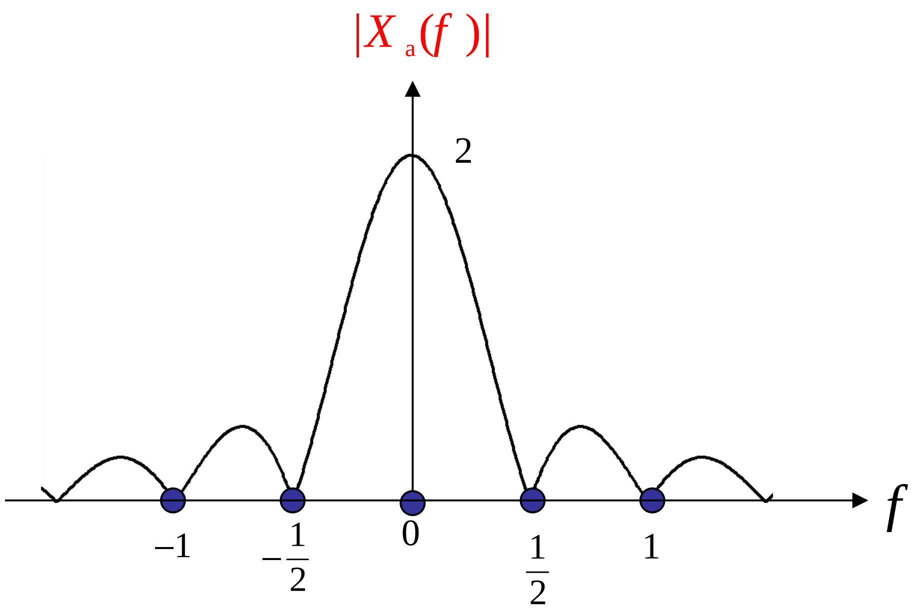


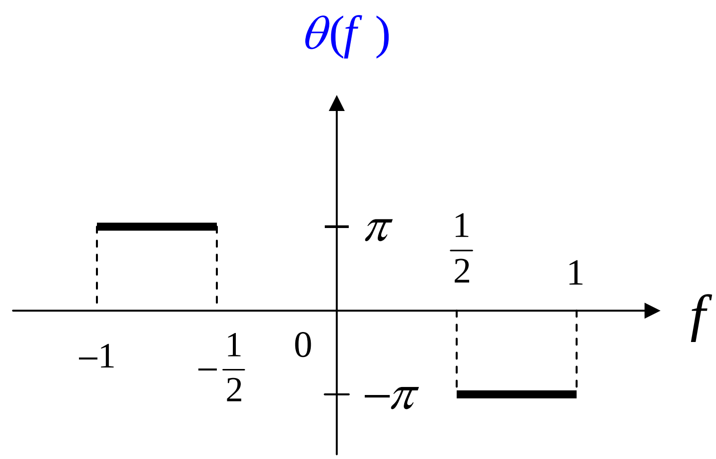


$\Leftrightarrow$ $X_a(f) = 2\text{sinc}(2f)$

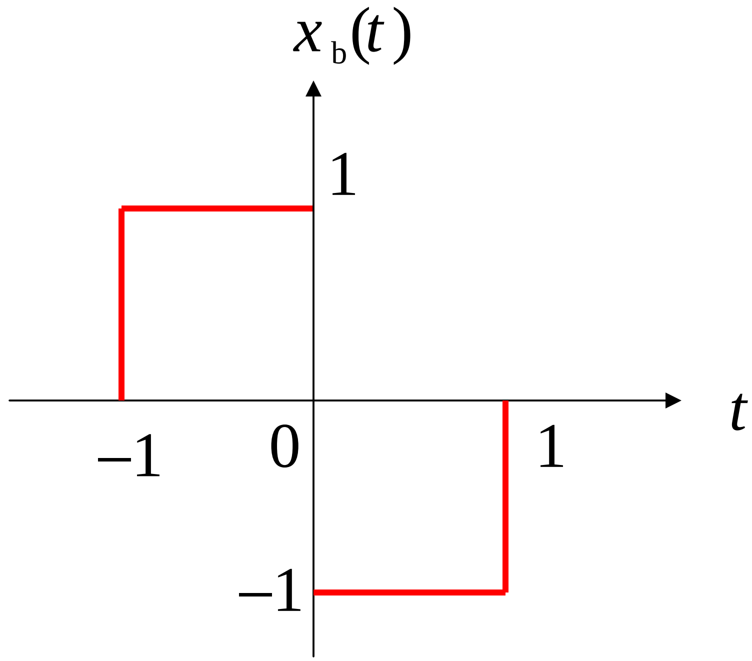


$$X_a(f) = |X_a(f)|e^{j\theta(f)}$$





Example 4-1 (continue)

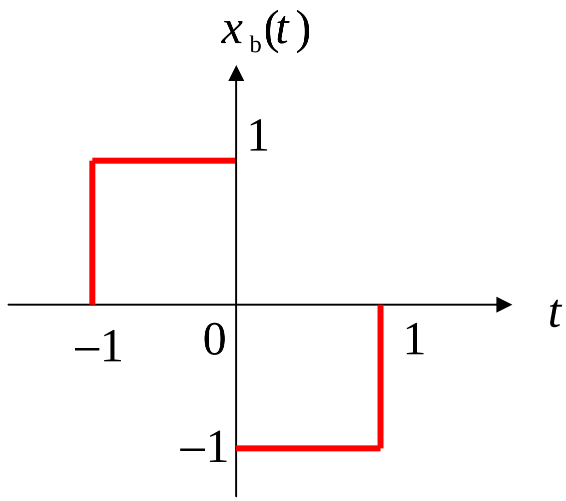


$$X_b(f) = \int_{-\infty}^{\infty} x_b(t) e^{-j2\pi ft} dt$$

$$= \int_{-1}^{0} (1)\, e^{-j2\pi ft} dt + \int_{0}^{1} (-1)\, e^{-j2\pi ft} dt = j2\pi f \,\mathrm{sinc}^2(2f)$$

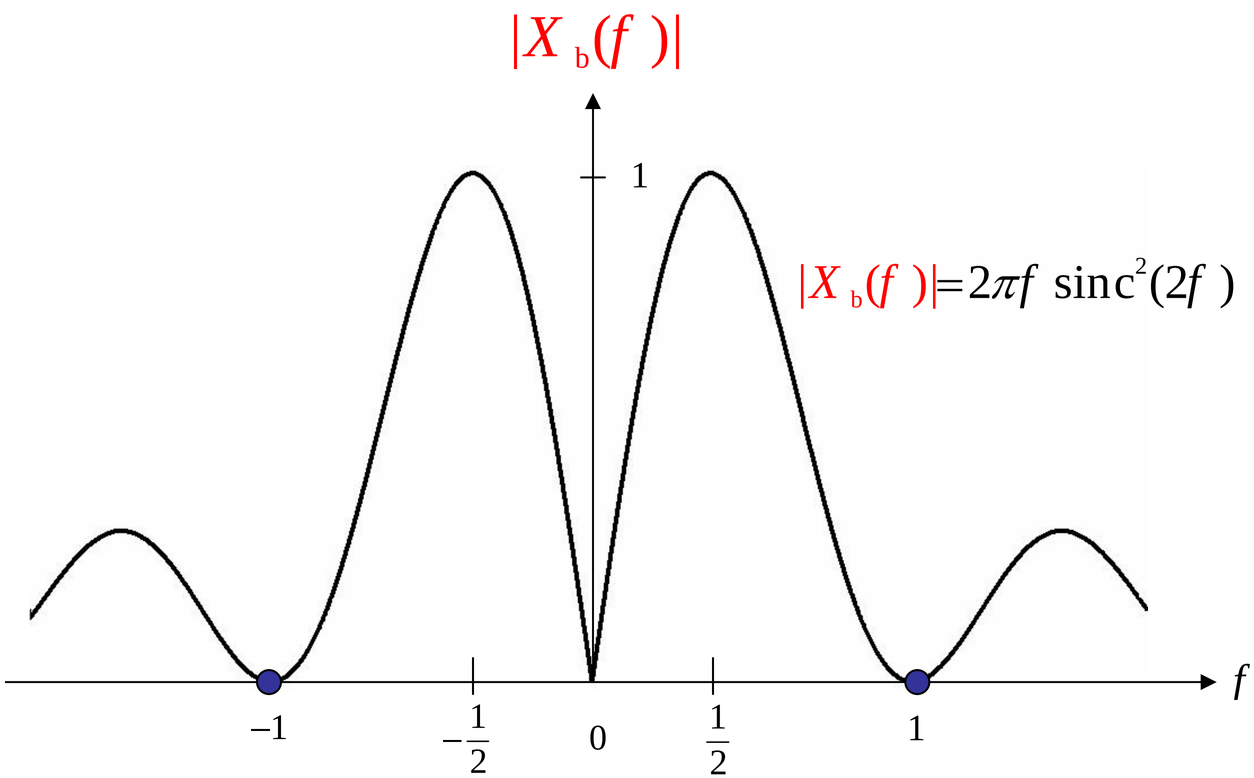
$$X_b(f) = |X_b(f)| e^{j\theta(f)}$$

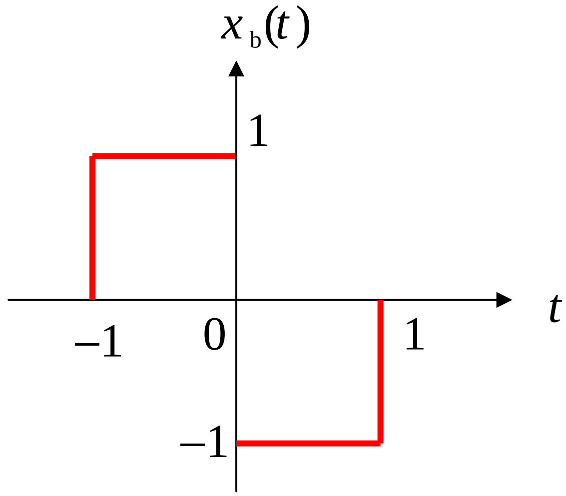
$$|X_b(f)| = 2\pi f \,\mathrm{sinc}^2(2f)$$



$$\Longleftrightarrow \quad X_b(f) = j\,2\pi f\ \mathrm{sinc}^2(2f)$$

$$X_b(f) = |X_b(f)|\,e^{j\theta(f)}$$





$$\Leftrightarrow \quad X_b(f) = j\,2\pi f\ \mathrm{sinc}^2(2f)$$

$$X_b(f) = |X_b(f)|\,e^{j\theta(f)}$$

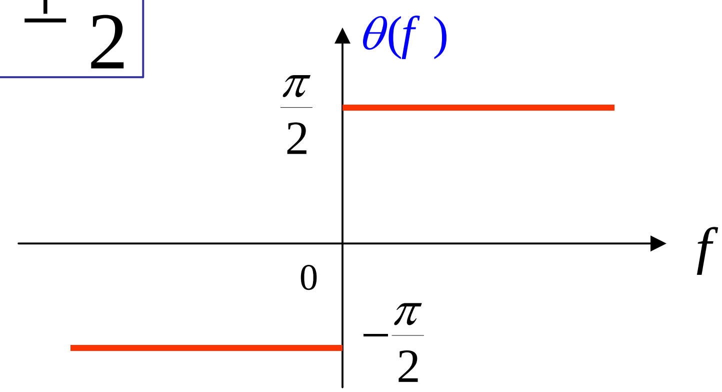
$$\frac{\pi}{2}$$

Always > 0 it add no angle (0^o)

$$X_b(f) = j\,2\pi f\ \ \mathrm{sinc}^2(2f)$$

When f < 0 it add no angle $\pm\dfrac{\pi}{2}$

When f > 0 it add no angle (0^o)



4.3 Energy Spectral Density

$$E \triangleq \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} x(t) x(t)^* \, dt$$

$$= \int_{-\infty}^{\infty} x(t)^* \left[\int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \right] dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t)^* X(f) e^{j2\pi ft} df \, dt$$

$$E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)^{*} X(f) e^{j2\pi ft} \, df \, dt$$

Reversal of the order of integration

$$E = \int_{-\infty}^{\infty} X(f) \left[\int_{-\infty}^{\infty} x(t)^{*} e^{j2\pi ft} \, dt \right] df$$

$$= \int_{-\infty}^{\infty} X(f) \left[\int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} \, dt \right]^{*} df$$

$$= \int_{-\infty}^{\infty} X(f) \left[\int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \right]^{*} df$$

$$\underbrace{\int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt}_{X^{*}(f)}$$

$$= \int_{-\infty}^{\infty} X(f) X^{*}(f) df = \int_{-\infty}^{\infty} |X(f)|^{2} df$$

$$E \triangleq \int_{-\infty}^{\infty} |x(t)|^{2} \, dt = \int_{-\infty}^{\infty} |X(f)|^{2} df$$

Note the similarity with Parsevals
For Fourier Series

$$\underbrace{P_{av} = \frac{1}{T_0} \int_{T_0} |x(t)|^{2} dt}_{\text{Time domain}} = \underbrace{\sum_{n=-\infty}^{\infty} |X_n|^{2}}_{\text{Frequency domain}}$$

$$E \triangleq \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

This is referred to as Parseval's Theorem for Fourier Transform

Define $G(f) \triangleq |X(f)|^2$

$$E = \int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} G(f) df$$

Energy (J)

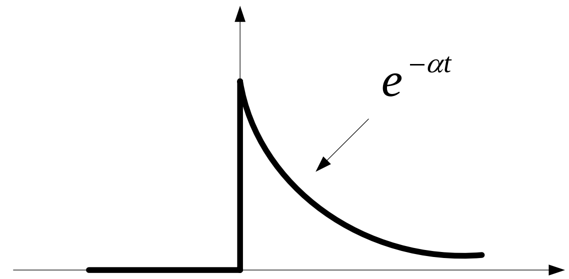
$\Rightarrow$ $G(f) = |X(f)|^2$ Is an **energy density** (J/Hz)

$$E_B = \int_{-B}^{B} G(f) df$$

Energy of the signal in the range $-B < f < B$

Example 4-2

$$x(t) = e^{-\alpha t} u(t) \qquad \alpha > 0$$



Find The Energy ?

Solution

Time Domain

$$E \triangleq \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} x^2(t) \, dt$$

$$= \int_0^{\infty} (e^{-\alpha t})^2 dt = \int_0^{\infty} e^{-2\alpha t} dt = \left. \frac{e^{-2\alpha t}}{-2\alpha} \right|_0^{\infty}$$

$$E = \int_{-\infty}^{\infty} x^2(t)\ dt = \left.\frac{e^{-2\alpha t}}{-2\alpha}\right|_0^{\infty} = \frac{e^{-2\alpha(\infty)} - e^{-2\alpha(0)}}{-2\alpha}$$

$$= \frac{0-1}{-2\alpha} = \frac{1}{2\alpha}$$

Frequency Domain

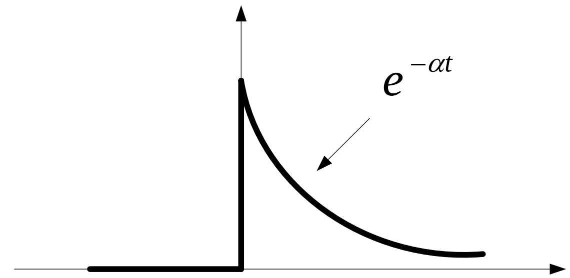
$$E=\int_{-\infty}^{\infty}\left|X(f)\right|^2 df=\int_{-\infty}^{\infty}G(f)\,df$$

We need ***X(f)***

$$X(f)=\int_{-\infty}^{\infty}x(t)e^{-j2\pi ft}dt=\int_{0}^{\infty}e^{-\alpha t}e^{-j2\pi ft}dt=\int_{0}^{\infty}e^{-(\alpha+j2\pi f)t}dt$$

$$=\frac{e^{-(\alpha+j2\pi f)t}}{-(\alpha+j2\pi f)}\Bigg|_0^{\infty}=\frac{\overbrace{e^{-(\alpha+j2\pi f)(\infty)}}^{0}-\overbrace{e^{-(\alpha+j2\pi f)(0)}}^{1}}{-(\alpha+j2\pi f)}=\frac{1}{(\alpha+j2\pi f)}$$

$$x(t) = e^{-\alpha t} u(t) \qquad \alpha > 0$$

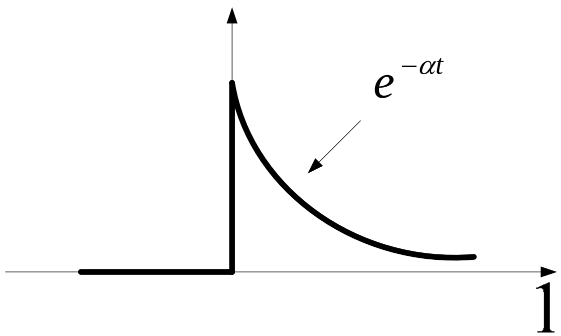


$$X(f) = \frac{1}{(\alpha + j2\pi f)}$$

4[th] entry in Fourier Transform Table
Table 4-2

TABLE 4-2
Fourier Transform Pairs

Pair Number	$x(t)$	$X(f)$
1.	$\Pi\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc} \tau f$
2.	$2W \operatorname{sinc} 2Wt$	$\Pi\left(\frac{f}{2W}\right)$
3.	$\Lambda\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc}^2 \tau f$
4.	$\exp(-\alpha t)u(t),\ \alpha > 0$	$\frac{1}{\alpha + j2\pi f}$



x(t)

$$x(t) = e^{-\alpha t} u(t) \qquad \alpha > 0 \quad \Longleftrightarrow \quad X(f) = \frac{1}{(\alpha + j2\pi f)}$$

$$E = \int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} G(f) df$$

$$G(f) \triangleq |X(f)|^2 = \left| \frac{1}{(\alpha + j2\pi f)} \right|^2 = \frac{1}{\alpha^2 + (2\pi f)^2}$$

$$E = \int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} \frac{df}{\alpha^2 + (2\pi f)^2}$$

$$x(t) = e^{-\alpha t} u(t) \qquad \alpha > 0 \quad \Longleftrightarrow \quad X(f) = \frac{1}{(\alpha + j2\pi f)}$$

$$E = \int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} \frac{df}{\alpha^2 + (2\pi f)^2}$$

Since the limits of the integration are $-\infty$ to ∞ which might cause $\dfrac{\infty}{\infty}$

Let

$$E_B = \int_{-B}^{-B} \frac{df}{\alpha^2 + (2\pi f)^2} \qquad \Rightarrow \qquad E = \lim_{B \to \infty} E_B$$

$$E_B = \int_{-B}^{-B} \frac{df}{\alpha^2 + (2\pi f)^2} \qquad E = \lim_{B \to \infty} E_B$$

Since $$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan \frac{x}{a}$$

Then $$E_B = \int_{-B}^{-B} \frac{df}{\alpha^2 + (2\pi f)^2} = \frac{1}{\pi \alpha} \tan^{-1} \frac{2\pi B}{\alpha}$$

$$\lim_{B \to \infty} \tan^{-1} \frac{2\pi B}{\alpha} = \frac{\pi}{2}$$

$$\Rightarrow \quad E = \lim_{B \to \infty} E_B = \frac{1}{2\alpha} = \int_{-\infty}^{\infty} |X(f)|^2 df$$

4-4 Fourier transform in the limit

Fourier transform pairs

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \quad \Longleftrightarrow \quad X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

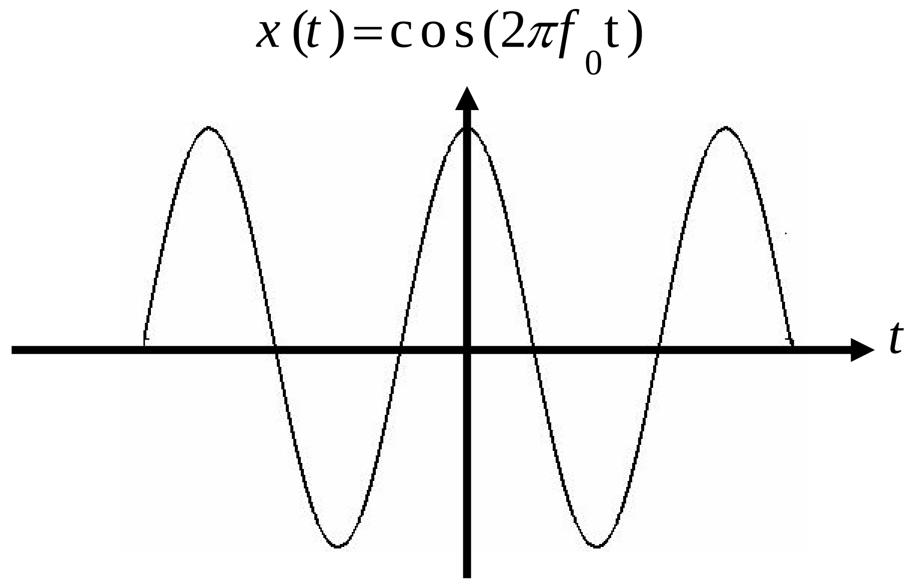
A condition for the existing of X(f) is the following

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

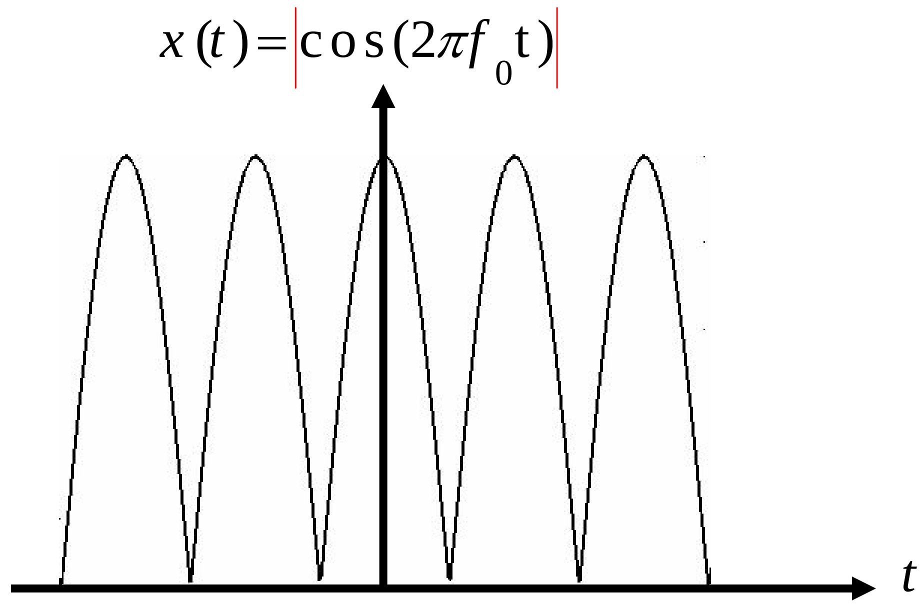
Absolutely Integrable

Example of function that is not absolutely integrable

$$x(t) = \cos(2\pi f_0 t)$$



The cosine is not Absolutely Integrable



$$\int_{-\infty}^{\infty} \left|\cos(2\pi f_0 t)\right| \, dt = \infty$$

4-5 Fourier Transform Theorems

The following theorems about Fourier Transform will be useful in obtaining additional Fourier Transform pairs

We will use the following notation for the Fourier Transform Pairs

$$x(t) \Leftrightarrow X(f)$$

(1) Linearity (superposition)

Let $x_1(t) \Leftrightarrow X_1(f)$ $\quad$ $x_2(t) \Leftrightarrow X_2(f)$

Then

$$x(t) = \{x_1(t) \pm x_2(t)\} \Leftrightarrow X(f) = \{X_1(t) \pm X_2(t)\}$$

$$x(t) = \{x_1(t) \pm x_2(t)\} \Leftrightarrow X(f) = \{X_1(f) \pm X_2(f)\}$$

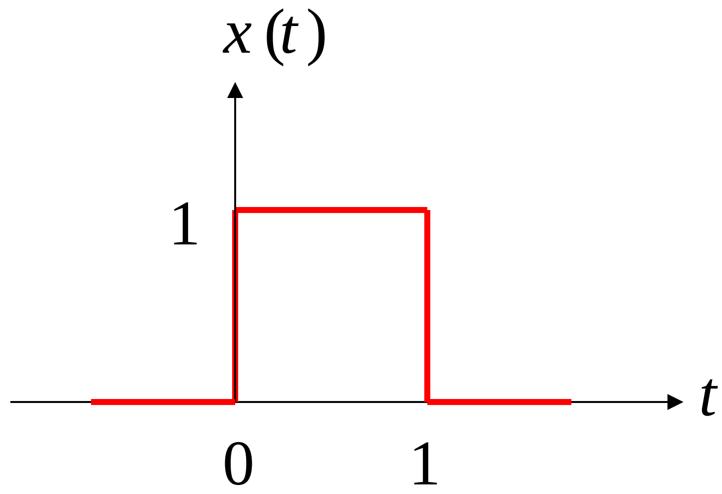
Proof

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} \{x_1(t) \pm x_2(t)\} e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{\infty} x_1(t) e^{-j2\pi ft} dt \pm \int_{-\infty}^{\infty} x_2(t) e^{-j2\pi ft} dt$$

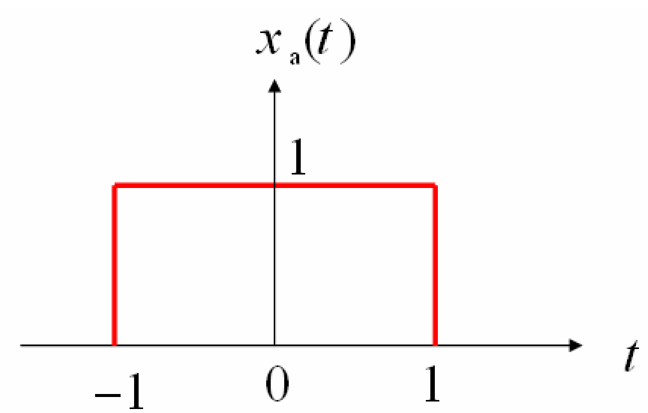
$$= X_1(f) \pm X_2(f)$$

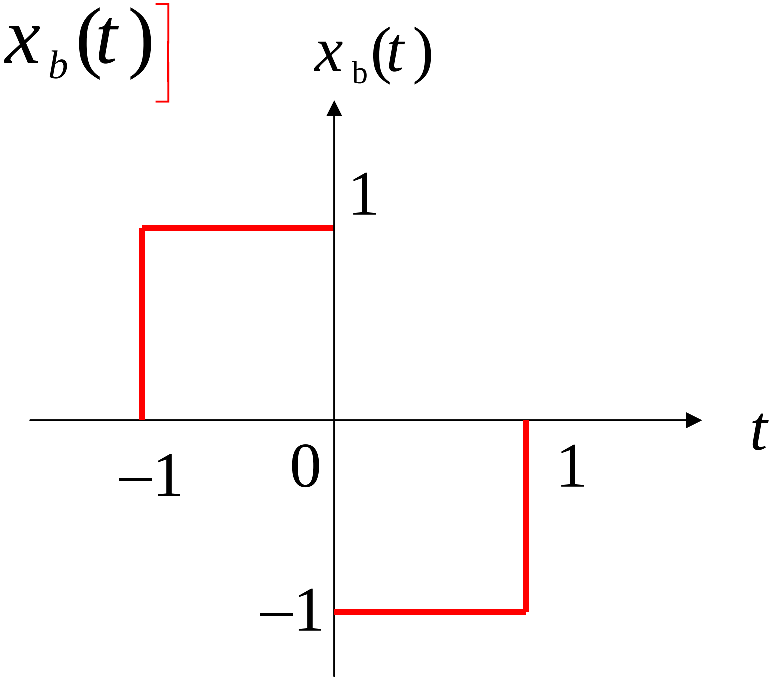
Example 4-3 Find the Fourier Transform of the pulse function



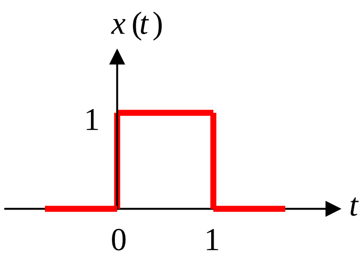
Solution

$$x(t) = \frac{1}{2}\left[x_a(t) - x_b(t)\right]$$

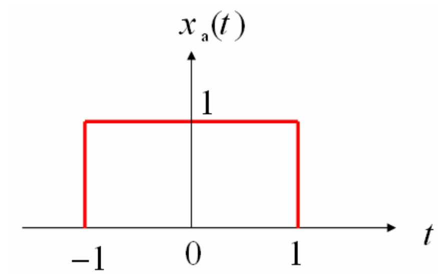




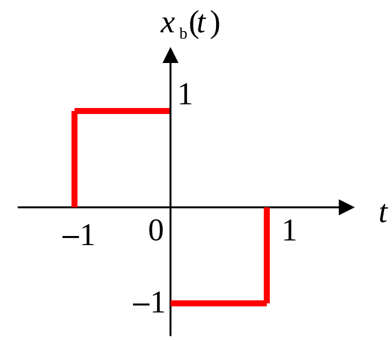
$$x(t) = \frac{1}{2}\left[x_a(t) - x_b(t)\right]$$



$$= \frac{1}{2}\Bigg\{$$



$$+$$



$$\Bigg\}$$

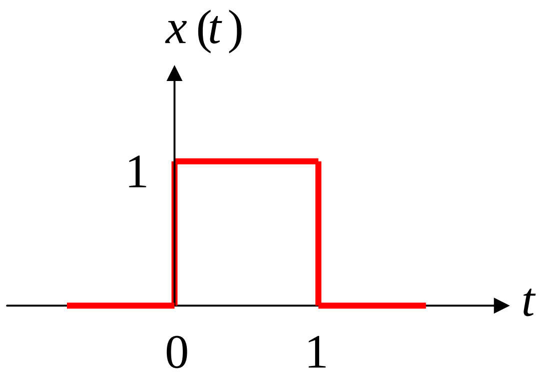
$$X(f) = \frac{1}{2}\left\{ X_a(f) - X_b(f) \right\}$$

From Example 4-1

$X_a(f)$: $2\text{sinc}(2f)$

$X_b(f)$: $j2\pi f\ \text{sinc}^2(2f)$

$$X(f) = \frac{1}{2}\left[2\text{sinc}(2f) - j2\pi f\ \text{sinc}^2(2f)\right]$$



$$x(t) = \frac{1}{2}\left[x_a(t) - x_b(t)\right]$$

$$X(f) = \frac{1}{2}\left\{ X_a(f) - X_b(f) \right\}$$

$$X(f) = \frac{1}{2}\left[2\operatorname{sinc}(2f) - j2\pi f\,\operatorname{sinc}^2(2f)\right] = \frac{1}{2}\left[2\frac{\sin(2\pi f)}{2\pi f} - j2\pi f\,\frac{\sin^2(\pi f)}{(\pi f)^2}\right]$$

$$= \frac{1}{2}\left[2\frac{\sin(2\pi f)}{2\pi f} - j\frac{1-\cos(2\pi f)}{2\pi f}\right] = \frac{1}{j2\pi f}\left[1-\cos(2\pi f) + j\sin(2\pi f)\right]$$

$$= \frac{1}{j2\pi f}\left[1 - e^{j2\pi f}\right] = \frac{e^{j\pi f}}{\pi f}\underbrace{\frac{e^{j\pi f} - e^{-j\pi f}}{2j}}_{\sin(\pi f)} = \frac{e^{j\pi f}}{\pi f}\sin(\pi f) = \frac{\sin(\pi f)}{\pi f}e^{j\pi f}$$

$$= \operatorname{sinc}(f)\, e^{j\pi f}$$

(2) Time-Delay Theorem

Let $x(t) \Leftrightarrow X(f)$ Then $x(t-t_0) \Leftrightarrow X(f)e^{-j2\pi f t_0}$

Proof Let $y(t)=x(t-t_0)$

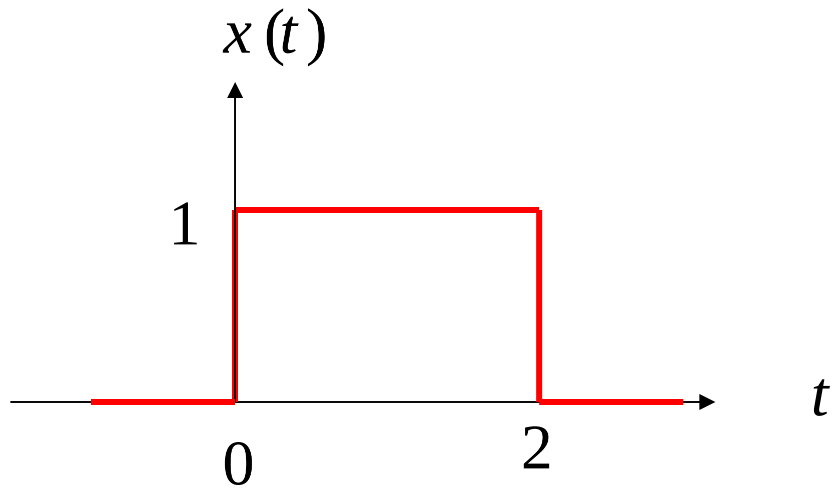
$$Y(f)=\int_{-\infty}^{\infty} y(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} x(t-t_0)e^{-j2\pi ft}dt$$

Change of variable $t'=t-t_0$ ⇒ $Y(f)=\int_{-\infty}^{\infty} x(t')e^{-j2\pi f(t'+t_0)}dt$

⇒ $Y(f)=\int_{-\infty}^{\infty} x(t')e^{-j2\pi ft'}e^{-j2\pi ft_0}dt = e^{-j2\pi ft_0}\underbrace{\int_{-\infty}^{\infty} x(t')e^{-j2\pi ft'}dt}_{X(f)}$

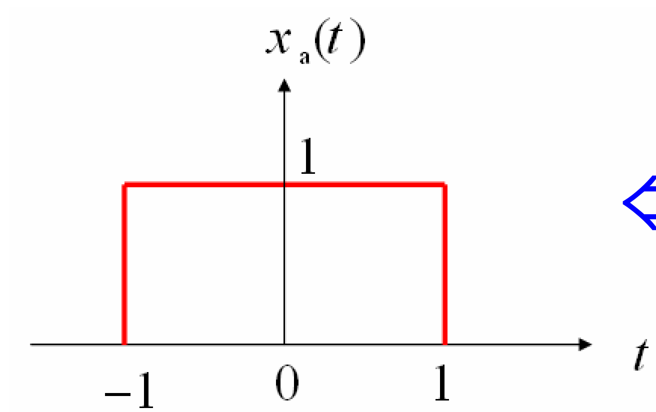
⇒ $x(t-t_0) \Leftrightarrow X(f)e^{-j2\pi ft_0}$

Example 4-4 Find the Fourier Transform of the pulse function



Solution

From Example 4-1



$$\Leftrightarrow \quad X_a(f) = 2\text{sinc}(2f)$$

Since $x(t) = x_a(t-1)$ ⇨

$$X(f) = 2\text{sinc}(2f)\, e^{-j2\pi f(1)}$$

$$= 2\text{sinc}(2f)\, e^{-j2\pi f}$$

(3) **Scale Change Theorem** (compressing or expanding)

Let $x(t) \Leftrightarrow X(f)$ Then $x(at) \Leftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$

Proof

Let $a > 0$ $\quad F\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-j2\pi ft} dt$

Change of variable $t' = at$ ⇨ $F\{x(at)\} = \int_{-\infty}^{\infty} x(t') e^{-j2\pi f\left(\frac{t'}{a}\right)} \frac{dt'}{a}$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(t') e^{-j2\pi f\left(\frac{t'}{a}\right)} dt'$$

$$F\{x(at)\} = \frac{1}{a}\int_{-\infty}^{\infty} x(t')e^{-j2\pi f\left(\frac{t'}{a}\right)}dt' = \frac{1}{a}\int_{-\infty}^{\infty} x(t')e^{-j2\pi\left(\frac{f}{a}\right)t'}dt'$$

Let $f' = \frac{f}{a}$ $\Rightarrow$ $F\{x(at)\} = \frac{1}{a}\underbrace{\int_{-\infty}^{\infty} x(t')e^{-j2\pi f' t'}dt'}_{X(f')}$

$\Rightarrow$ $F\{x(at)\} = \frac{1}{a}X(f') = \frac{1}{a}X\left(\frac{f}{a}\right)$

Now Let $a < 0 \rightarrow at = -|a|t$

$$\Rightarrow F\{x(at)\} = \int_{-\infty}^{\infty} x(-|a|t)e^{-j2\pi ft}dt$$

Change of variable $t' = -|a|t \Rightarrow dt = \frac{dt'}{-|a|}$ $t = \infty \rightarrow t' = -\infty$, $t = -\infty \rightarrow t' = \infty$

$$\Rightarrow F\{x(at)\} = \int_{\infty}^{-\infty} x(t')e^{-j2\pi f\left(-\frac{t'}{|a|}\right)}\frac{dt'}{-|a|} = \frac{1}{|a|}\underbrace{\int_{-\infty}^{\infty} x(t')e^{-j2\pi\left(-\frac{f}{|a|}\right)t'}dt'}_{X\left(-\frac{f}{|a|}\right)}$$

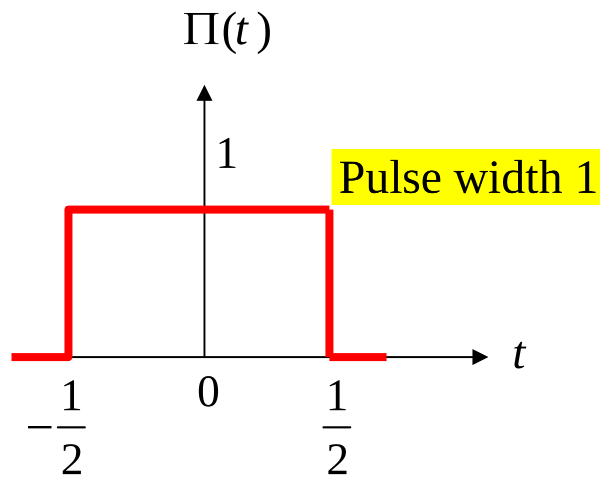
$$F\{x(at)\} = \frac{1}{|a|}X\left(-\frac{f}{|a|}\right)$$

Since $a < 0 \rightarrow -|a| = a \Rightarrow$

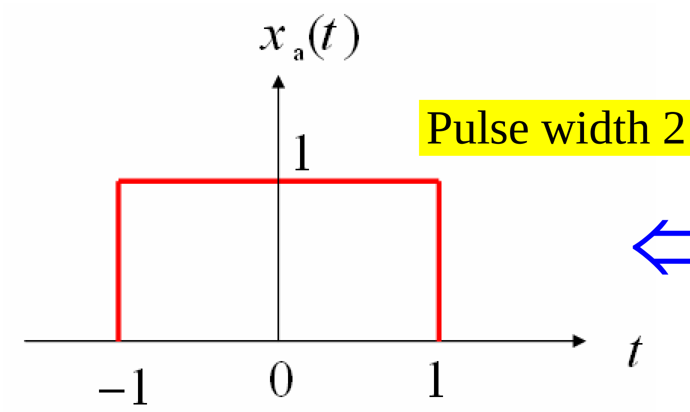
$$F\{x(at)\} = \frac{1}{|a|}X\left(\frac{f}{|a|}\right)$$

Example 4-6

Since the pulse function was defined in chapter 1



From Example 4-1

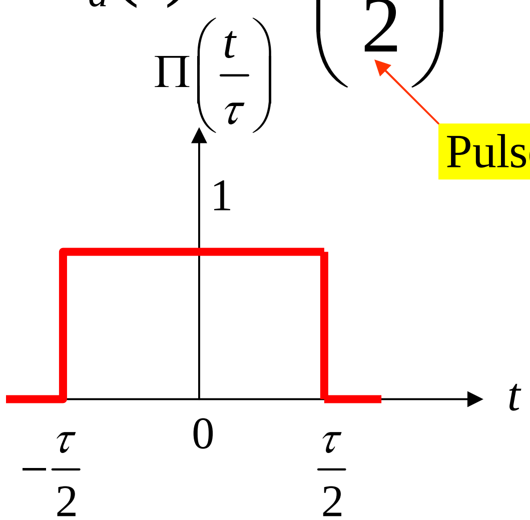


$$\Leftrightarrow \quad X_a(f) = 2\,\mathrm{sinc}(2f)$$

$$x_a(t) = \Pi\left(\frac{t}{2}\right) \Leftrightarrow X_a(f) = 2\,\mathrm{sinc}(2f)$$

Pulse width 2

Now for pulse width τ



$$\Leftrightarrow \quad \tau\,\mathrm{sinc}(\tau f)$$

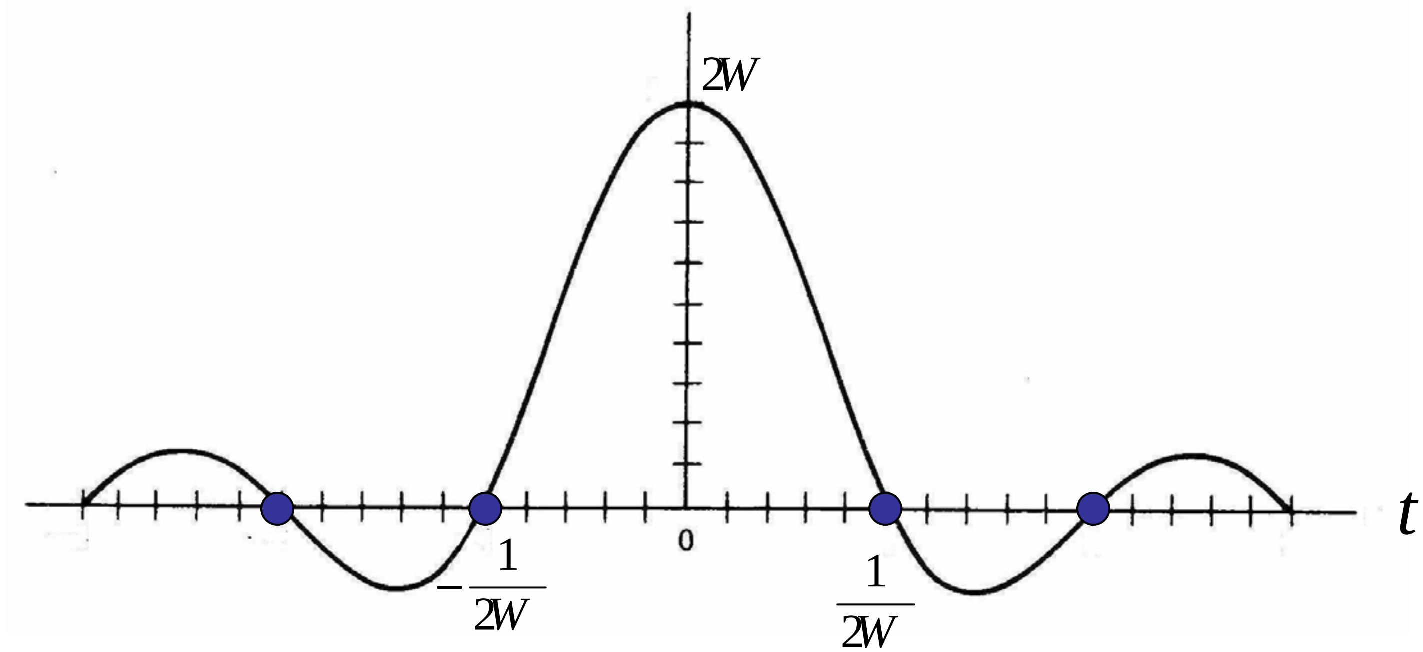
1st entry in Fourier Transform Table 4-2

(4) Duality ازدواجية

If $x(t) \Leftrightarrow X(f)$ then $X(t) \Leftrightarrow x(-f)$

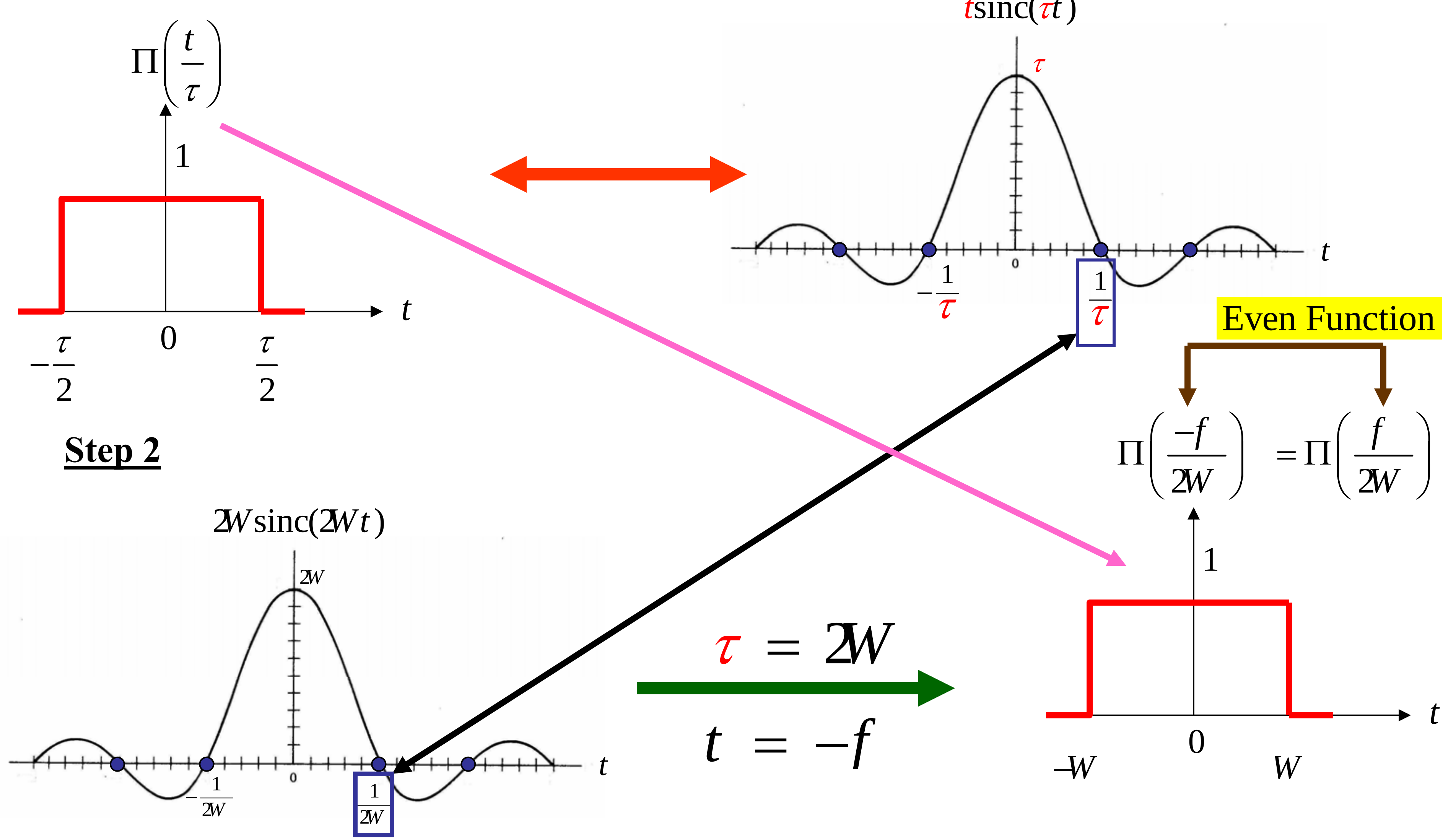
Example 4-7 Find the F.T of $\left\{2W\operatorname{sinc}(2Wt)\right\}$

$$2W\operatorname{sinc}(2Wt)$$



$$X(t) \Leftrightarrow x(-f)$$

Step 1 from Known transform from the F.T Table



(4) Frequency Translation Theorem

Let $x(t) \Leftrightarrow X(f)$ Then $x(t)e^{j2\pi f_0 t} \Leftrightarrow X(f - f_0)$

Proof

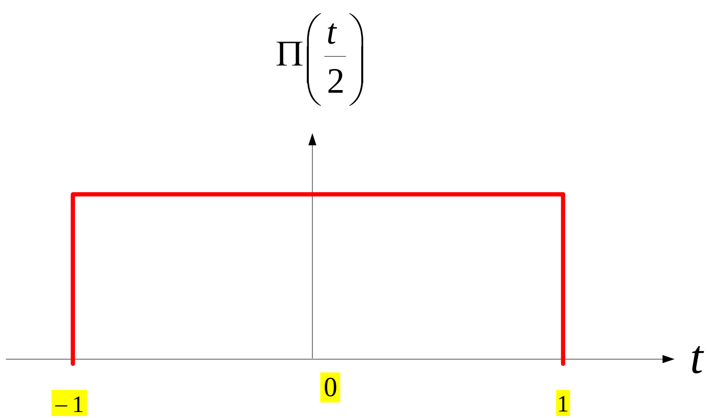
$$F\left\{x(t)e^{j2\pi f_0 t}\right\} = \int_{-\infty}^{\infty} x(t)e^{j2\pi f_0 t}e^{-j2\pi f t}dt = \int_{-\infty}^{\infty} x(t)e^{-j2\pi(f - f_0)t}dt$$

Change of variable $f' = f - f_0$

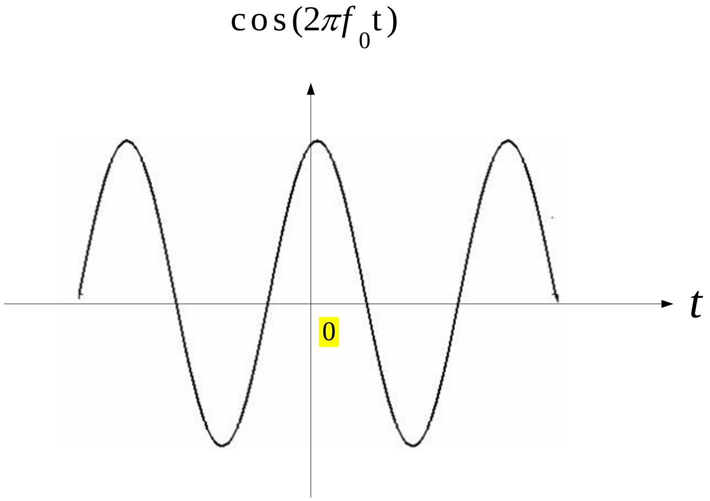
$$\Rightarrow \quad F\left\{x(t)e^{j2\pi f_0 t}\right\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi f' t}dt = X(f') = X(f - f_0)$$

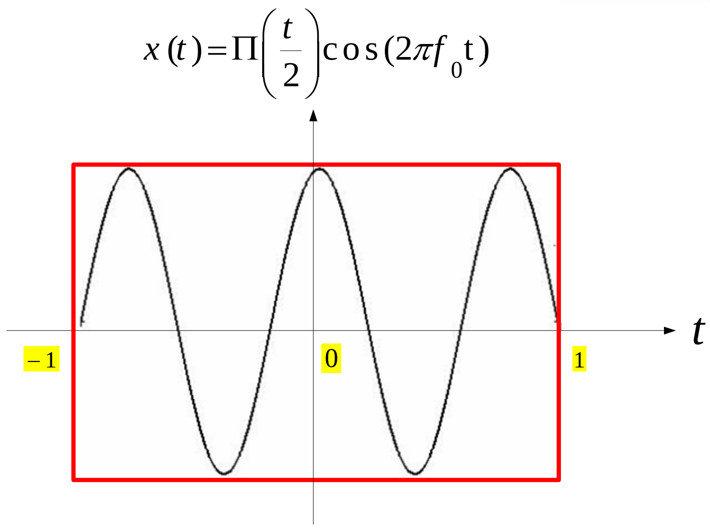
Example 4-8 Find the Fourier Transform of the function

$$x(t) = \Pi\left(\frac{t}{2}\right)\cos(20\pi t)$$

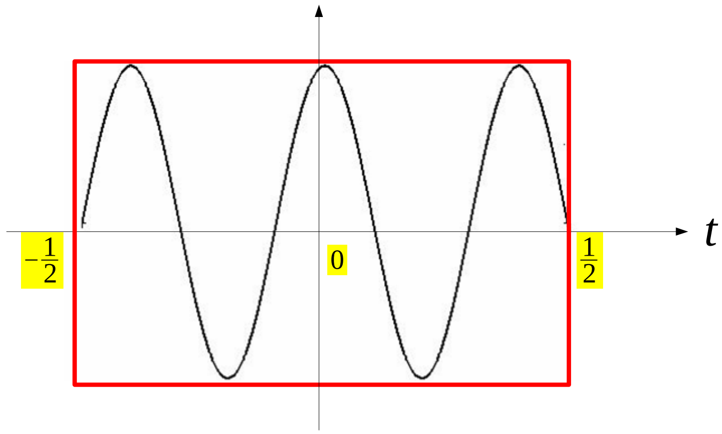


X





$x(t)=\Pi\left(\frac{t}{2}\right)\cos(2\pi f_0 t)$



$$x(t)=\Pi\left(\frac{t}{2}\right)\cos(2\pi f_0 t)$$

$$= \Pi\left(\frac{t}{2}\right)\left\{\frac{e^{2\pi f_0 t} - e^{-2\pi f_0 t}}{2}\right\}$$

$$x(t)= \frac{1}{2}\Pi\left(\frac{t}{2}\right)e^{2\pi f_0 t} - \frac{1}{2}\Pi\left(\frac{t}{2}\right)e^{-2\pi f_0 t}$$

Since $x(t)e^{\pm j2\pi f_0 t} \Leftrightarrow X(f \mp f_0)$ **and** $\Pi\left(\frac{t}{2}\right) \Leftrightarrow 2\text{sinc}(2f)$

Therefore

$$x(t)=\Pi\left(\frac{t}{2}\right)\cos(2\pi f_0 t) \Leftrightarrow \text{sinc}(2(f-f_0))+\text{sinc}(2(f+f_0))$$

(5) Differentiation and Integration Theorem

Differentiation

$$F\left[\frac{dx(t)}{dt}\right]=\int_{-\infty}^{\infty}\frac{dx(t)}{dt}e^{-j2\pi ft}dt$$

Using integration by parts $\int_a^b u\,dv = uv\Big|_a^b-\int_a^b v\,du$

$$u=e^{-j2\pi ft}$$

$$du=-j2\pi fe^{-j2\pi ft}$$

$$dv=\frac{dx(t)}{dt}dt=dx(t)$$

$$v=x(t)$$

$$\Rightarrow\quad F\left[\frac{dx(t)}{dt}\right]=x(t)e^{-j2\pi ft}\Big|_{-\infty}^{\infty}+j2\pi f\int_{-\infty}^{\infty}x(t)e^{-j2\pi ft}dt$$

$$F\left[\frac{dx(t)}{dt}\right] = x(t)e^{-j2\pi ft}\Big|_{-\infty}^{\infty} + j2\pi f\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

$$= \left\{x(\infty)e^{-j2\pi f(\infty)} - x(-\infty)e^{-j2\pi f(-\infty)}\right\} + j2\pi f\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

Since $x(t)$ is absolutely integrable $\int_{-\infty}^{\infty} |x(t)|dt < \infty \Rightarrow$ $x(\infty) \to 0$, $x(-\infty) \to 0$

$$\Rightarrow \left\{x(\infty)e^{-j2\pi f(\infty)} - x(-\infty)e^{-j2\pi f(-\infty)}\right\} \longrightarrow 0$$

$$\Rightarrow F\left[\frac{dx(t)}{dt}\right] = j2\pi f\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt = j2\pi f\ X(f)$$

$$\Rightarrow \frac{dx(t)}{dt} \Leftrightarrow j2\pi f\ X(f)$$

$$\frac{d^n x(t)}{dt^n} \Leftrightarrow (j2\pi f)^n X(f)$$

Integration

$$\int_{-\infty}^{t} x(t')dt' \Leftrightarrow \frac{1}{j2\pi f}X(f) + \frac{1}{2}X(0)\delta(f)$$

Example Find the Fourier Transform of the unit step function $u(t)$

$$\delta(t) \Leftrightarrow 1$$

$$u(t) = \int_{-\infty}^{t} \delta(t')dt' \Leftrightarrow \frac{1}{j2\pi f}(1) + \frac{1}{2}(1)\Big|_{f=0}\delta(f)$$

$$u(t) \Leftrightarrow \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

(6) The convolution Theorem

$$x_1(t) * x_2(t) \Leftrightarrow X_1(f)X_2(f)$$

Convolution in Time ⇔ **Multiplication in Frequency**

Proof

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda)x_2(t-\lambda)d\lambda$$

$$x_2(t) = \int_{-\infty}^{\infty} X_2(f)e^{j2\pi ft}df \Rightarrow x_2(t-\lambda) = \int_{-\infty}^{\infty} X_2(f)e^{j2\pi f(t-\lambda)}df$$

$$= \int_{-\infty}^{\infty} X_2(f)e^{j2\pi ft}e^{-j2\pi f\lambda}df$$

$$x_1(t)*x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda)x_2(t-\lambda)d\lambda$$

$$x_2(t)=\int_{-\infty}^{\infty} X_2(f)e^{j2\pi ft}df = \int_{-\infty}^{\infty} X_2(f)e^{j2\pi ft}e^{-j2\pi f\lambda}df$$

Now substitute $x_2(t)$ (as the inverse Fourier Transform) in the convolution integral

$$x_1(t)*x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda)\left\{\int_{-\infty}^{\infty} X_2(f)e^{j2\pi ft}e^{-j2\pi f\lambda}df\right\}d\lambda$$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda) \left\{ \int_{-\infty}^{\infty} X_2(f) e^{j2\pi f t} e^{-j2\pi f \lambda} df \right\} d\lambda$$

Exchanging the order of integration , we have

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} X_2(f) \underbrace{\left\{ \int_{-\infty}^{\infty} x_1(\lambda) e^{-j2\pi f \lambda} d\lambda \right\}}_{X_1(f)} e^{j2\pi f t} df$$

$$x_1(t) * x_2(t) = \underbrace{\int_{-\infty}^{\infty} X_1(f) X_2(f) e^{j2\pi f t} df}_{\text{Inverse Fourier Transform}}$$

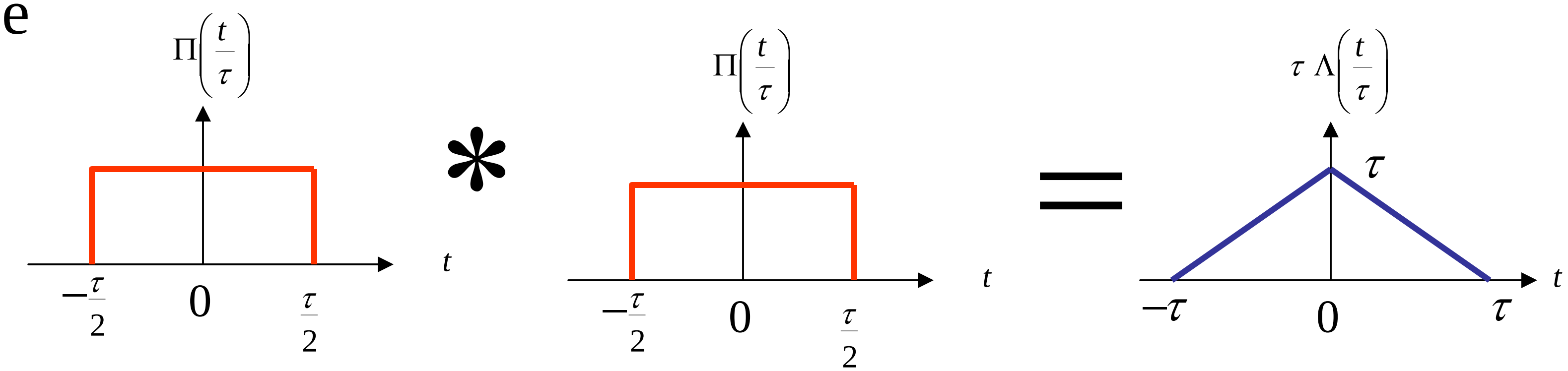
$$\Rightarrow \quad x_1(t) * x_2(t) \Leftrightarrow X_1(f) X_2(f)$$

Example 4-13 Find the Fourier Transform of following

$$\Pi\left(\frac{t}{\tau}\right) * \Pi\left(\frac{t}{\tau}\right)$$

Solution

Since



$$\mathrm{FT}\left\{\Pi\left(\frac{t}{\tau}\right) * \Pi\left(\frac{t}{\tau}\right)\right\} = \mathrm{FT}\left\{\Pi\left(\frac{t}{\tau}\right)\right\} . \mathrm{FT}\left\{\Pi\left(\frac{t}{\tau}\right)\right\}$$

$$= \tau \,\mathrm{sinc}(\tau f) \bullet \tau \,\mathrm{sinc}(\tau f) \quad = \tau^2 \mathrm{sinc}^2(\tau f)$$

(7) The multiplication Theorem

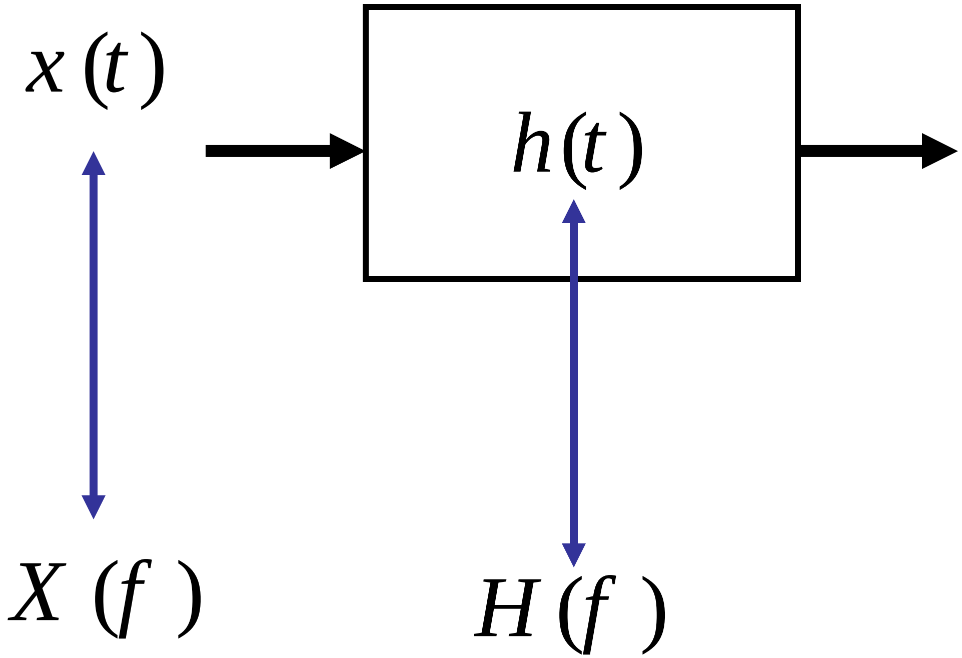
$$x_1(t)x_2(t) \Leftrightarrow X_1(f) * X_2(f)$$

Proof

Similar to the convolution theorem , left as an exercise

Example 4-15

4-6 System Analysis with Fourier Transform



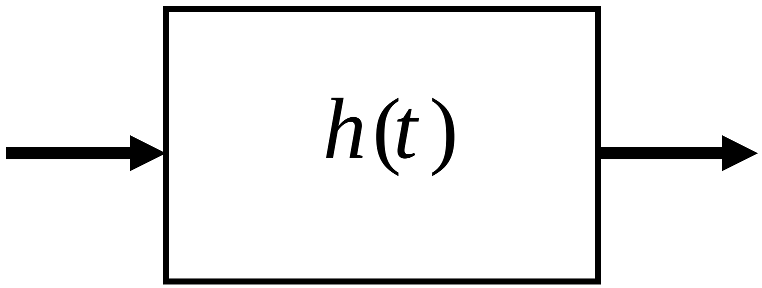
$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\lambda) h(t-\lambda)\, d\lambda$$

$$Y(f) = X(f) H(f)$$

$$y(t) = x(t) * h(t) \Longleftrightarrow Y(f) = X(f) H(f)$$

$x(t)$

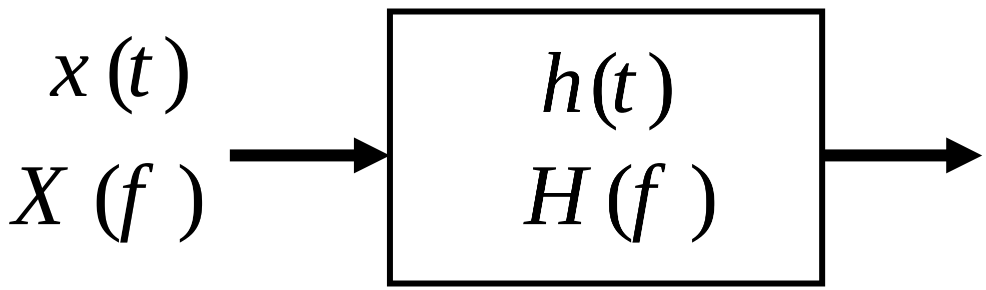


$y(t) = x(t) * h(t)$

$X(f)$ $\quad$ $H(f)$ $\quad$ $Y(f) = X(f)H(f)$

$$H(f) = \frac{Y(f)}{X(f)} \triangleq \text{Transfer Function}$$

$h(t)$ (Impulse Response) $\Leftrightarrow$ $H(f)$ (Transfer Function)



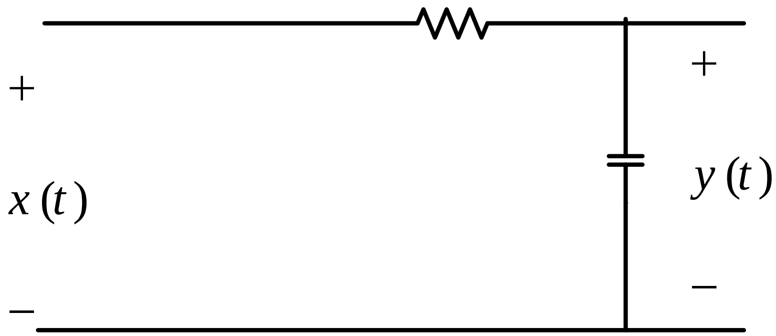
$$y(t) = x(t) * h(t)$$
$$Y(f) = X(f)H(f)$$

$$H(f) = |H(f)|\, e^{j\angle H(f)}$$

Since $h(t)$ is real

$$|H(f)| = |H(-f)| \qquad \text{Even}$$
$$\angle H(f) = -\angle H(-f) \qquad \text{Odd}$$

Example 4-16 Find the Transfer Function for the following *RC* circuit



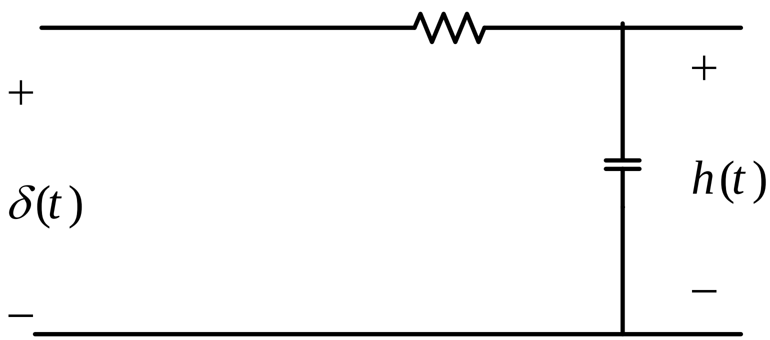
$$RC\frac{dy(t)}{dt} + y(t) = x(t) \qquad -\infty < t < \infty$$

R

Method 1

From Example 2-10 (Chapter 2) we can find *h*(t) by solving differential equation as follows

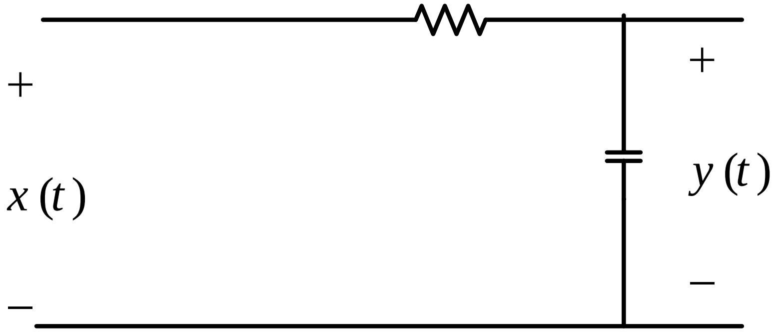
C



$$RC\frac{dh(t)}{dt} + h(t) = \delta(t)$$

$$h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}u(t)$$

Method 2



$$RC\frac{dy(t)}{dt}+y(t)=x(t)$$

R

We will find *h*(t) using Fourier Transform Method rather than solving differential equation as follows

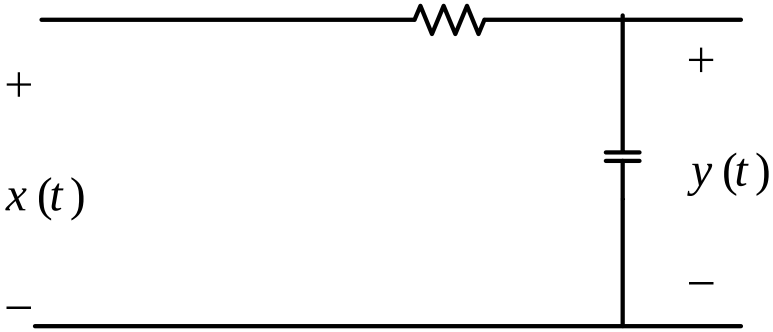
C

$$\mathrm{FT}\left[RC\frac{dy(t)}{dt}+y(t)\right]=\mathrm{FT}\left[x(t)\right]$$

$$RC(j2\pi f)Y(\mathrm{f})+Y(\mathrm{f})=X(\mathrm{f})$$

$$\left[(j2\pi fRC)+1\right]Y(\mathrm{f})=X(\mathrm{f})$$

$$H(f)=\frac{Y(f)}{X(f)}=\frac{1}{(j2\pi fRC)+1}$$



$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

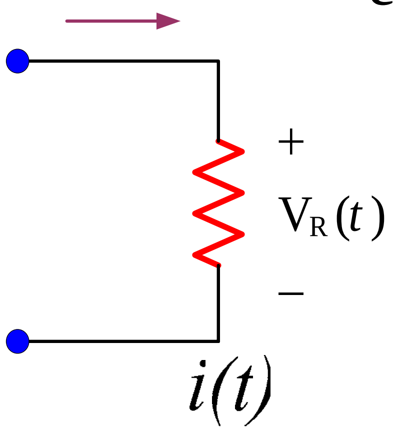
$$H(f) = \frac{Y(f)}{X(f)} = \frac{1}{(j2\pi fRC) + 1} = \frac{(1/RC)}{(j2\pi f) + (1/RC)}$$

From Table 4-2

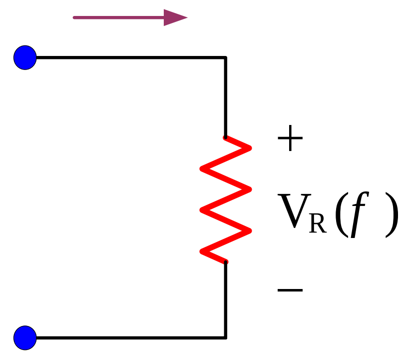
$$x(t) = e^{-\alpha t}u(t) \quad \alpha > 0 \quad \Leftrightarrow \quad \frac{1}{\alpha + j2\pi f}$$

$$h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}u(t)$$

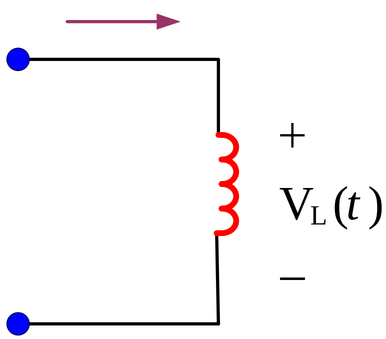
Method 3 In this method we are going to transform the circuit to the Fourier domain . However we first see the FT on Basic elements



$$V_R(t) = R\,i(t)$$

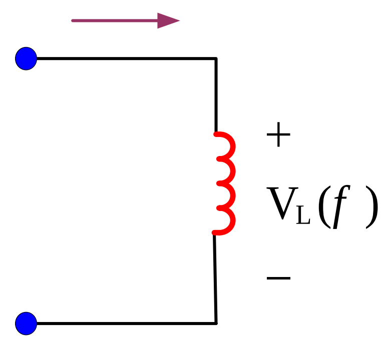


$$V_R(f) = R\,I(f)$$

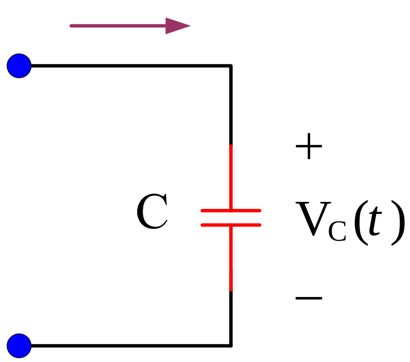


$$V_L(t) = L\frac{di(t)}{dt}$$

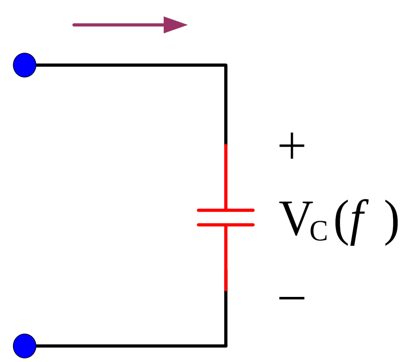
R



$$V_L(f) = L(j2\pi f)I(f)$$
$$= (j2\pi fL)I(f)$$



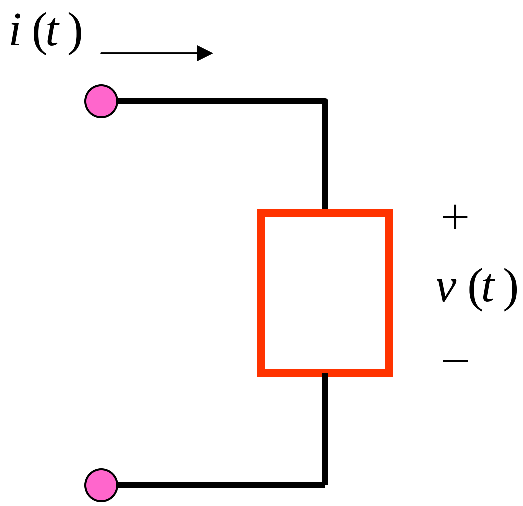
$$i(t) = C\frac{dV_C(t)}{dt}$$

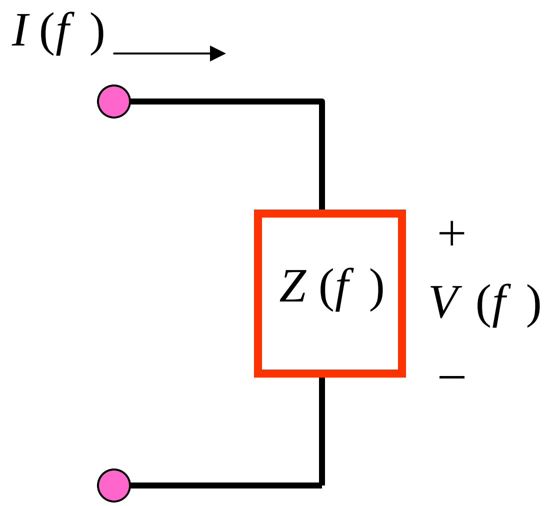


$$I(f) = C(j2\pi f)V_C(f)$$

$$V_C(f) = \frac{1}{(j2\pi fC)}I(f)$$

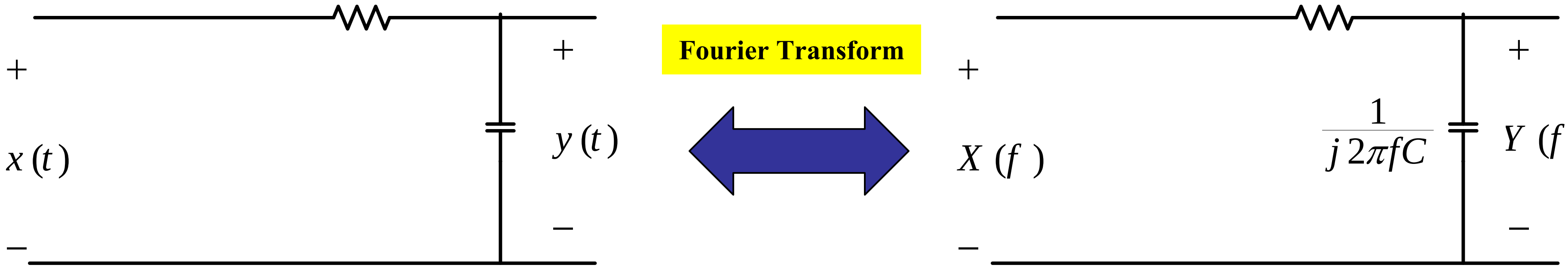
Method 3





$$Z(f) = \begin{cases} R & \text{Resistor} \\ j2\pi fL & \text{Inductor} \\ \dfrac{1}{j2\pi fC} & \text{Capacitor} \end{cases}$$

Method 3

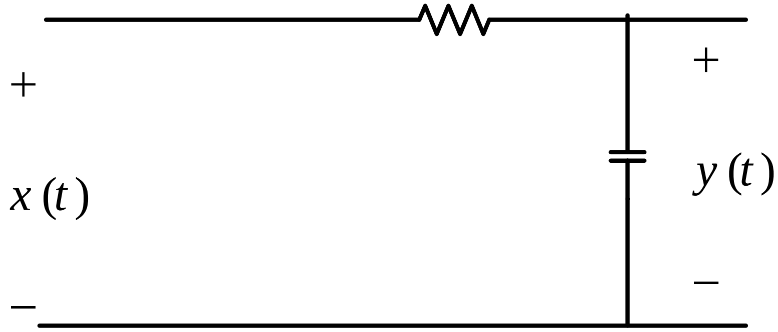


R

$$Y(f) = \frac{\frac{1}{j2\pi fC}}{R + \frac{1}{j2\pi fC}} X(f) = \frac{1}{j2\pi fRC + 1} X(f)$$

C

$$\frac{Y(f)}{X(f)} = \frac{1}{j2\pi fRC + 1} = H(f) \quad \Rightarrow \quad h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$$



$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

$$H(f) = \frac{Y(f)}{X(f)} = \frac{1}{1+(j2\pi fRC)}$$

$$H(f) = \frac{1}{1+\left(j\dfrac{f}{\dfrac{1}{2\pi RC}}\right)} = \frac{1}{1+\left(j\dfrac{f}{f_3}\right)}$$

were $f_3 = \dfrac{1}{2\pi RC}$

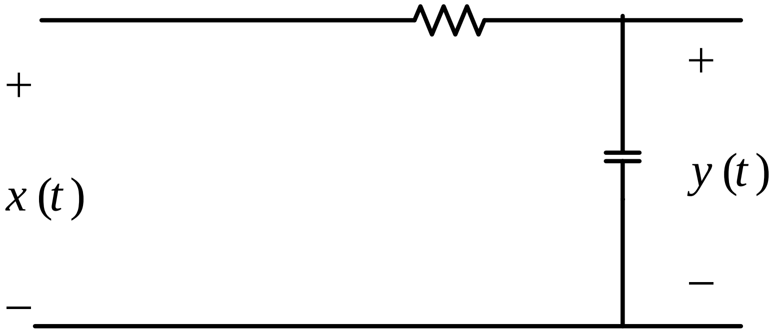
$$\omega_3 = \frac{1}{RC}$$

at $f = f_3$

$$H(f_3) = \frac{1}{1+\left(j\dfrac{f_3}{f_3}\right)} = \frac{1}{1+j}$$

$$|H(f_3)| = \frac{1}{\sqrt{2}} = 0.707$$

$$\angle H(f) = -\tan^{-1}(1) = \frac{\pi}{4}$$

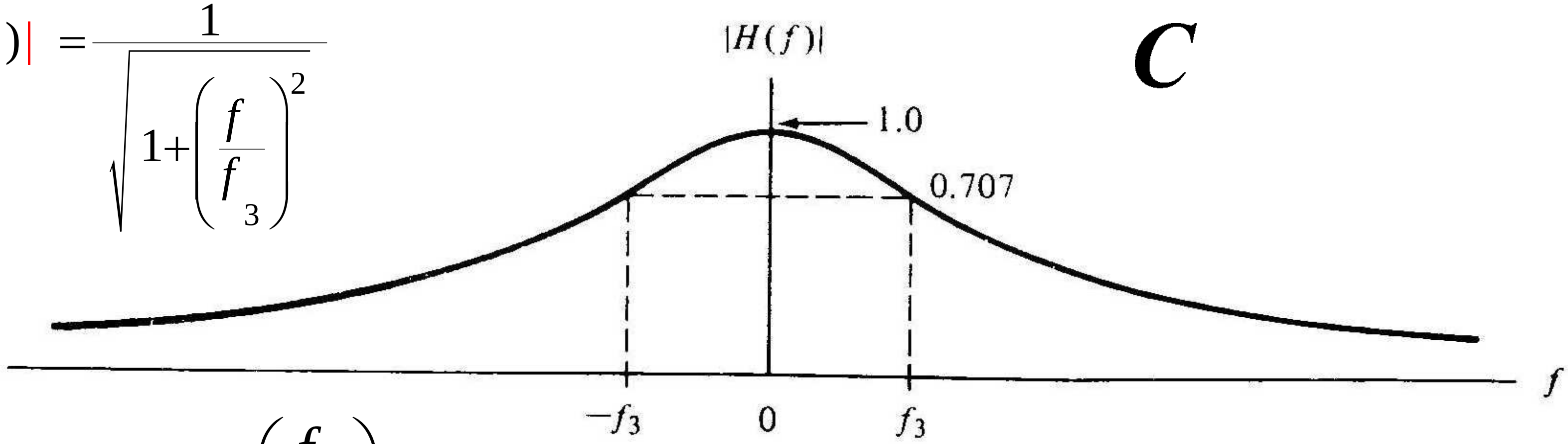


$$H(f) = \frac{1}{1 + \left(j \frac{f}{f_3} \right)}$$

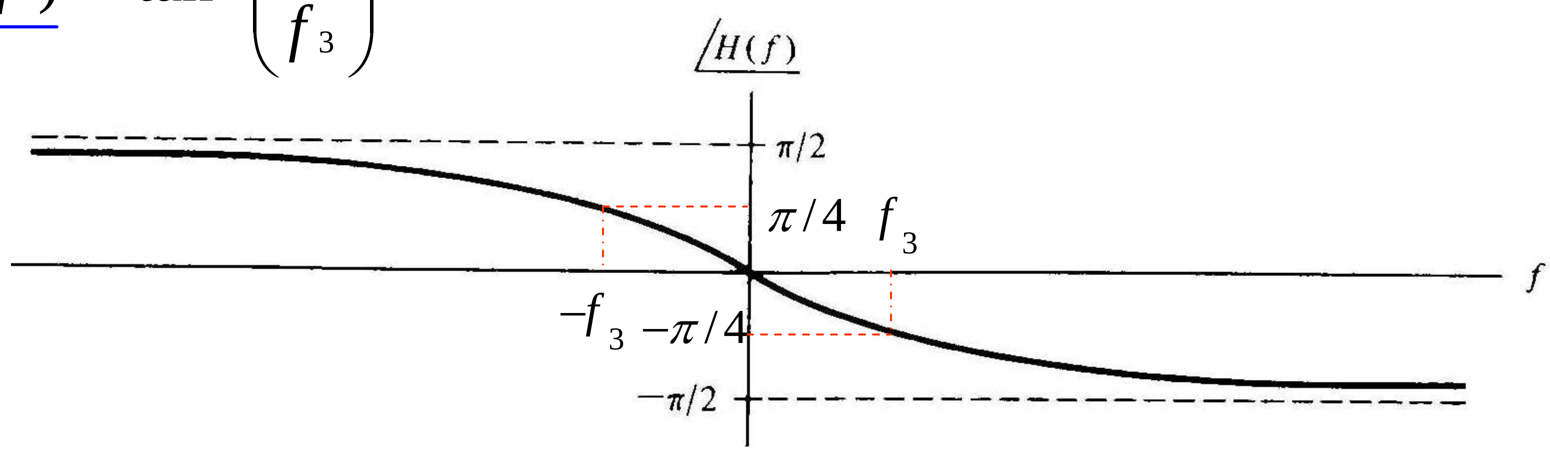
R

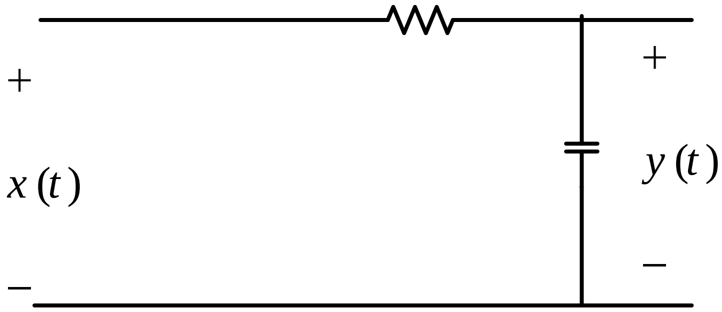
$$|H(f)| = \frac{1}{\sqrt{1 + \left(\frac{f}{f_3} \right)^2}}$$

C



$$\angle H(f) = -\tan^{-1}\left(\frac{f}{f_3} \right)$$





Find y(t) if the input x(t) is

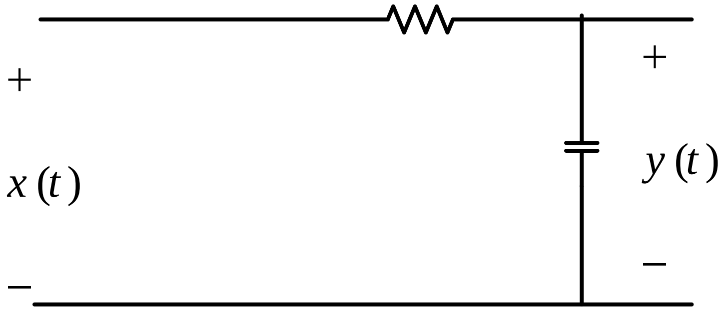
$$x(t) = A e^{-\alpha t} u(t)$$

Solution 1

Using the time domain (convolution method , Chapter 2)

$$y(t) = x(t) * h(t)$$

Find h(t) as was done before $h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$



$$x(t) = A e^{-\alpha t} u(t)$$

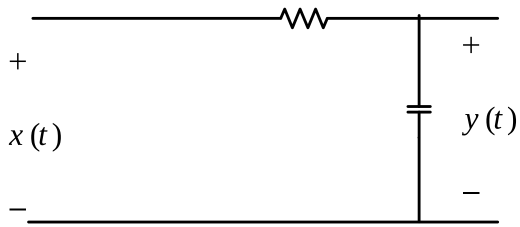
$$h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$$

Solution 2 Using the Frequency domain (Chapter 4)

$$y(t) = x(t) * h(t) \Longleftrightarrow Y(f) = X(f)H(f)$$

$$X(f) = \frac{1}{\alpha + j2\pi f} \qquad H(f) = \frac{(1/RC)}{(j2\pi f) + (1/RC)}$$

$$Y(f) = \frac{A/RC}{(\alpha + j2\pi f)(1/RC + j2\pi f)}$$



Find y(t) if the input x(t) is

$$x(t) = A e^{-\alpha t} u(t) \qquad X(f) = \frac{1}{\alpha + j2\pi f}$$

$$h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t) \qquad H(f) = \frac{(1/RC)}{(j2\pi f) + (1/RC)}$$

$$Y(f) = X(f)H(f) = \frac{A/RC}{(\alpha + j2\pi f)(1/RC + j2\pi f)}$$

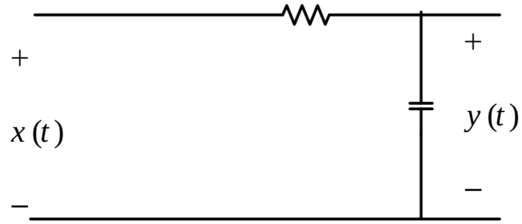
For $\alpha RC \neq 1$ Using partial fraction expansion

$$Y(f) = \frac{A}{\alpha RC - 1}\left[\frac{1}{1/RC + j2\pi f} - \frac{1}{\alpha + j2\pi f}\right]$$

From Table 4-2

$$y(t) = \frac{A}{\alpha RC - 1}\left[e^{-t/RC} - e^{-\alpha t}\right] u(t)$$

$$x(t) = A e^{-\alpha t} u(t)$$



$$Y(f) = \frac{A/RC}{(\alpha + j2\pi f)(1/RC + j2\pi f)}$$

R

For $\alpha RC \neq 1$ $\quad y(t) = \frac{A}{\alpha RC - 1}\left[e^{-t/RC} - e^{-\alpha t}\right] u(t)$ **C**

For $\alpha RC = 1$ $\quad \lim_{\alpha RC \to 1} y(t) = \lim_{\alpha \to \frac{1}{RC}} y(t) = \frac{A}{1-1}\left[e^{-t/RC} - e^{-\alpha/RC}\right] = \frac{0}{0}$

Using L'Hôpital's rule $\quad \lim_{x \to \alpha} \frac{f(x)}{g(x)} = \frac{0}{0} \Rightarrow \lim_{x \to \alpha} \frac{\frac{df(x)}{dx}}{\frac{dg(x)}{dx}}$

$$\lim_{\alpha \to \frac{1}{RC}} y(t) = \lim_{\alpha \to \frac{1}{RC}} A \frac{\frac{d}{d\alpha}\left[e^{-t/RC} - e^{-\alpha t}\right]}{\frac{d}{d\alpha}(\alpha RC - 1)} u(t) = \lim_{\alpha \to \frac{1}{RC}} A \frac{-(-t)e^{-\alpha t}}{RC} u(t)$$

$$= A\left(\frac{t}{RC}\right) e^{-t/RC} u(t)$$

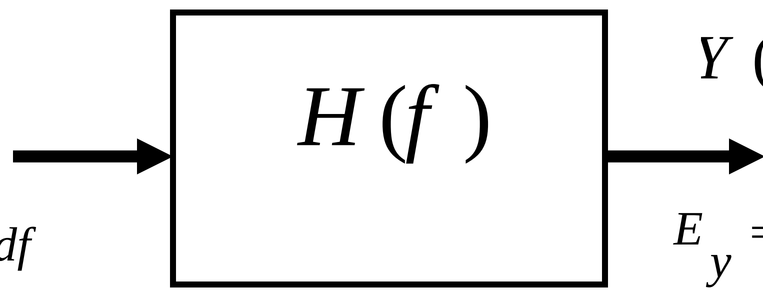
Energy input output relation

Recall Parseval's Theorem for Fourier Transform

$$E \triangleq \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} G(f) df$$

were $G(f) \triangleq |X(f)|^2$ Is an **energy density** (J/Hz)

$X(f)$



$Y(f) = X(f)H(f)$

$$E_x = \int_{-\infty}^{\infty} G_x(f) df$$

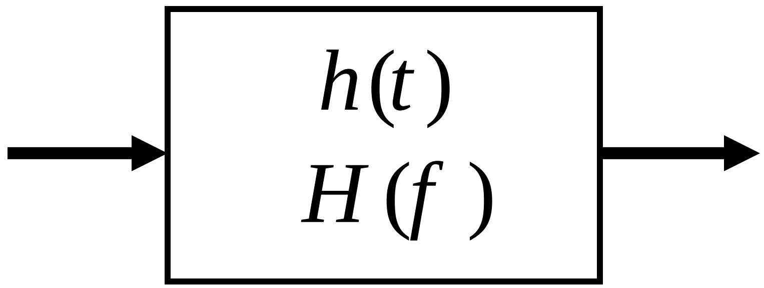
$$E_y = \int_{-\infty}^{\infty} G_y(f) df = \int_{-\infty}^{\infty} |Y(f)|^2 df$$

$$= \int_{-\infty}^{\infty} |X(f)H(f)|^2 df = \int_{-\infty}^{\infty} |X(f)|^2 |H(f)|^2 df$$

$$G_y(f) = |X(f)|^2 |H(f)|^2 = G_x(f) \, |H(f)|^2$$

$x(t)$

$X(f)$



$$y(t) = x(t) * h(t)$$

$$Y(f) = X(f)H(f)$$

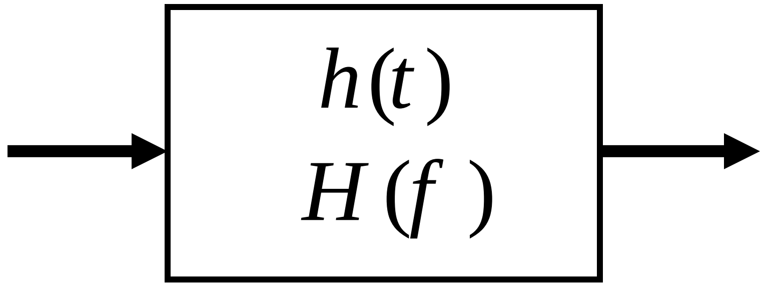
$G_x(f)$

$$G_y(f) = G_x(f)\,|H(f)|^2$$

4-7 Steady-state system Response to Sinusoidal Inputs by Means of Fourier Transform

$x(t)$

$X(f)$

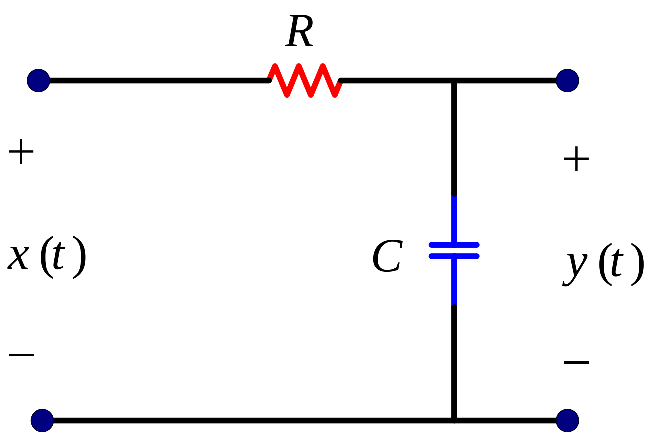


$$y(t) = x(t) * h(t)$$

$$Y(f) = X(f)H(f)$$

$$X(f) = |X(f)|\, e^{j\angle X(f)} \qquad H(f) = |H(f)|\, e^{j\angle H(f)}$$

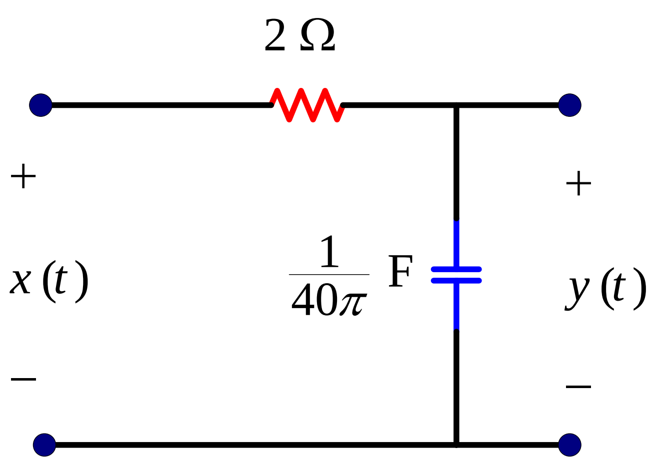
$$\Rightarrow \quad Y(f) = |X(f)|\,|H(f)|\, e^{j\angle X(f) + \angle H(f)}$$



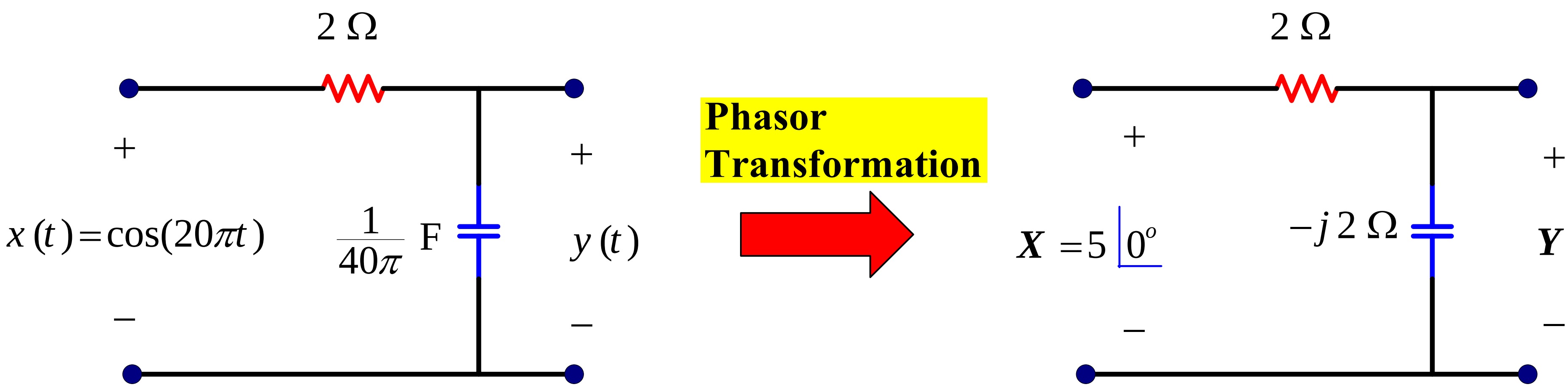
$$RC\frac{dy(t)}{dt} + y(t) = x(t) \qquad -\infty < t < \infty$$

$$H(f) = \frac{1}{1 + j2\pi fRC}$$

if $x(t) = \cos(20\pi t)$ find y(t) for the circuit shown ?



Method 1 Phasor method

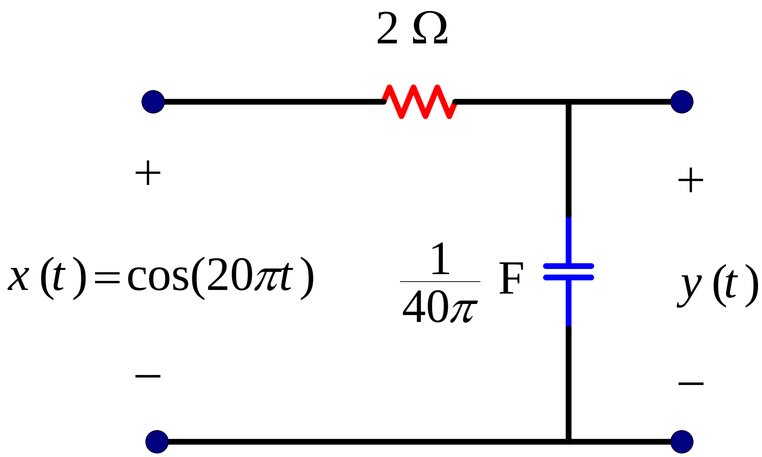


Voltage Division

$$\boldsymbol{Y} = \frac{-j2}{2-j2}\, 5\angle 0^o \quad = \frac{5}{\sqrt{2}} \angle -45^o \ \text{V}$$

$$\text{y(t)} = \frac{5}{\sqrt{2}} \cos(20\pi t - 45^o) \ \text{V}$$

Method 2 Fourier Transform method



$$RC\frac{dy(t)}{dt}+y(t)=x(t) \quad -\infty<t<\infty$$

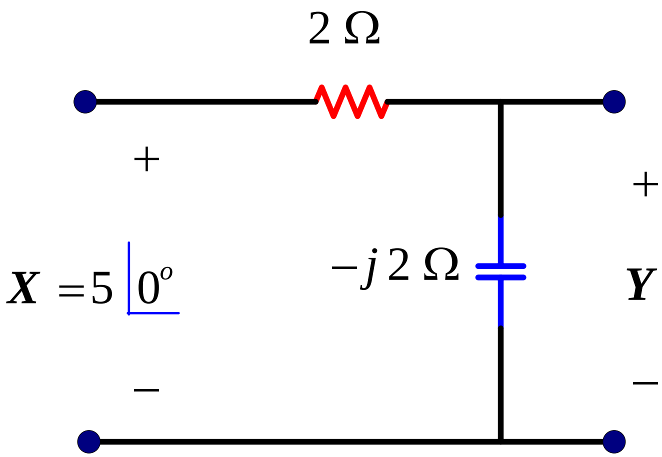
$$H(f)=\frac{1}{1+j2\pi fRC}$$

$$\Rightarrow H(f)=\frac{1}{1+j2\pi f(2)\left(\frac{1}{40\pi}\right)}=\frac{10}{10+jf}$$

$$X(f)=\frac{5}{2}\delta(f-10)+\frac{5}{2}\delta(f+10)$$

$$Y(f)=X(f)H(f)=\left[\frac{5}{2}\delta(f-10)+\frac{5}{2}\delta(f+10)\right]\frac{10}{1+jf}$$

$$=\frac{25}{10+jf}\delta(f-10)+\frac{25}{10+jf}\delta(f+10)$$



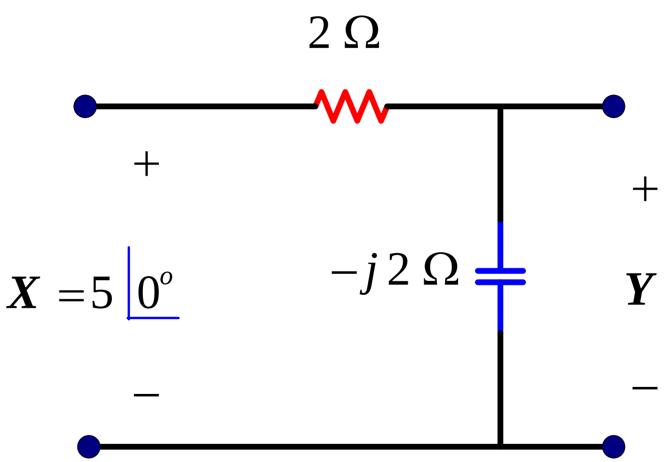
$$Y(f) = \frac{25}{10+jf}\delta(f-10) + \frac{25}{10+jf}\delta(f+10)$$

$$= \frac{25}{10+j(10)}\delta(f-10) + \frac{25}{10+j(-10)}\delta(f+10)$$

$$= \frac{25}{10\left(\sqrt{2}\angle 45^o\right)}\delta(f-10) + \frac{25}{10\left(\sqrt{2}\angle -45^o\right)}\delta(f+10)$$

$$= \frac{5}{2\sqrt{2}\angle 45^o}\delta(f-10) + \frac{5}{2\sqrt{2}\angle -45^o}\delta(f+10)$$

$$= \frac{5}{2\sqrt{2}}e^{-j45^o}\delta(f-10) + \frac{5}{2\sqrt{2}}e^{j45^o}\delta(f+10)$$

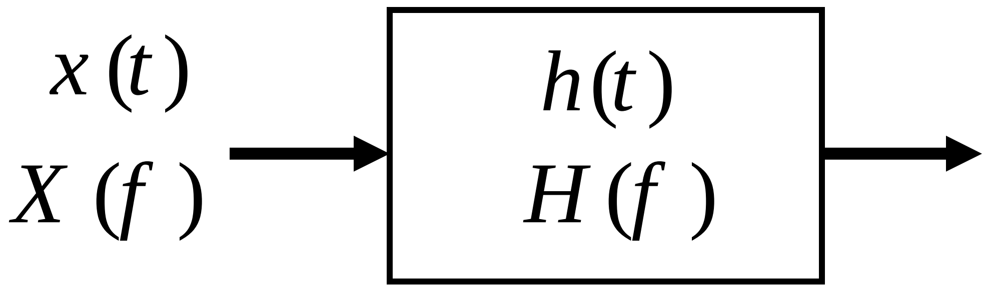


$$Y(f) = \frac{5}{2\sqrt{2}} e^{-j45^o} \delta(f - 10) + \frac{5}{2\sqrt{2}} e^{j45^o} \delta(f + 10)$$

constants with respect to *f*

$$y(t) = \frac{5}{2\sqrt{2}} e^{-j45^o} e^{j2\pi(10)t} + \frac{5}{2\sqrt{2}} e^{j45^o} e^{-j2\pi(10)t}$$

$$y(t) = \frac{5}{2\sqrt{2}} e^{j(20\pi t - 45^o)} + \frac{5}{2\sqrt{2}} e^{-j(20\pi t - 45^o)} = \frac{5}{\sqrt{2}} \cos(20\pi t - 45^o) \text{ V}$$



$$y(t) = x(t) * h(t)$$

$$Y(f) = X(f)H(f)$$

$$H(f) = |H(f)|\, e^{j\angle H(f)}$$

Let x(t) be a periodical signal

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t} \quad \text{were} \quad X_n = |X_n|\, e^{j\angle X_n}$$

Fourier Transform

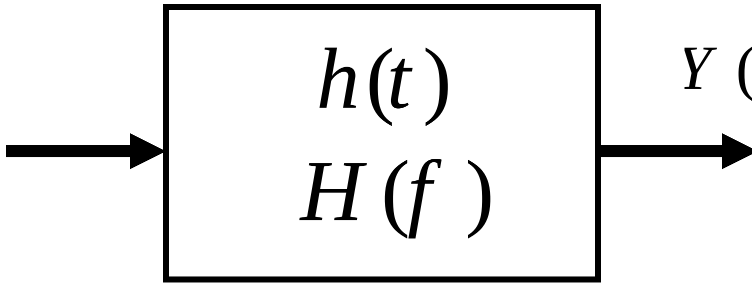
$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t} \Longleftrightarrow X(f) = \sum_{n=-\infty}^{\infty} X_n \delta(f - nf_0)$$

$$Y(f) = X(f)H(f) = H(f) \sum_{n=-\infty}^{\infty} X_n \delta(f - nf_0)$$

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t}$$

$$y(t) = x(t) * h(t)$$

$$X_n = |X_n| \, e^{j\angle X_n}$$



$$Y(f) = X(f)H(f)$$

$$X(f) = \sum_{n=-\infty}^{\infty} X_n \delta(f - nf_0)$$

$$H(f) = |H(f)| \, e^{j\angle H(f)}$$

$$Y(f) = X(f)H(f) = H(f)\sum_{n=-\infty}^{\infty} X_n \delta(f - nf_0)$$

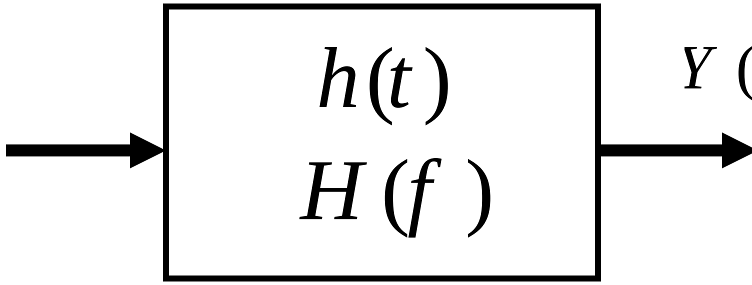
$$= \sum_{n=-\infty}^{\infty} X_n H(f)\delta(f - nf_0)$$

$$= \sum_{n=-\infty}^{\infty} X_n H(nf_0)\delta(f - nf_0)$$

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t}$$

$$X_n = |X_n| e^{j\angle X_n}$$

$$X(f) = \sum_{n=-\infty}^{\infty} X_n \delta(f - nf_0)$$



$$y(t) = x(t) * h(t)$$

$$Y(f) = X(f) H(f)$$

$$H(f) = |H(f)| e^{j\angle H(f)}$$

$$Y(f) = \sum_{n=-\infty}^{\infty} X_n H(nf_0) \delta(f - nf_0)$$

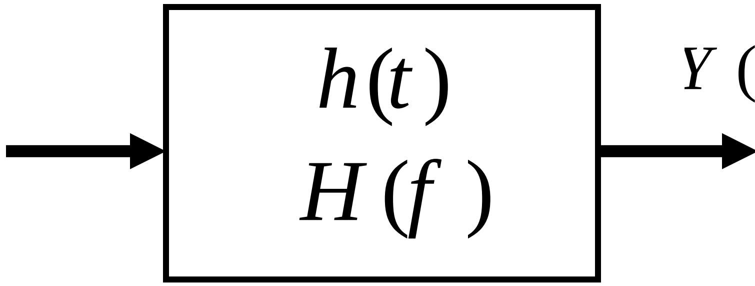
$$= \sum_{n=-\infty}^{\infty} |X_n| e^{j\angle X_n} \, |H(nf_0)| e^{j\angle H(nf_0)} \delta(f - nf_0)$$

$$= \sum_{n=-\infty}^{\infty} |X_n| \, |H(nf_0)| e^{j\left(\angle X_n + \angle H(nf_0)\right)} \delta(f - nf_0)$$

$$x(t)=\sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t}$$

$$X_n = |X_n|\, e^{j\angle X_n}$$

$$X(f)=\sum_{n=-\infty}^{\infty} X_n \delta(f-nf_0)$$



$$y(t) = x(t) * h(t)$$

$$Y(f)=X(f)H(f)$$

$$H(f) = |H(f)|\, e^{j\angle H(f)}$$

$$Y(f)=\sum_{n=-\infty}^{\infty} |X_n|\, |H(nf_0)|\, e^{j\left(\angle X_n + \angle H(nf_0)\right)} \quad \delta(f-nf_0)$$

Constant with respect to Fourier Transform

$$y(t)=\sum_{n=-\infty}^{\infty} |X_n|\, |H(nf_0)|\, e^{j\left(\angle X_n + \angle H(nf_0)\right)} \quad e^{j2\pi n f_0}$$

$$y(t)=\sum_{n=-\infty}^{\infty} |X_n|\, |H(nf_0)| \quad e^{j\left(2\pi n f_0 + \angle X_n + \angle H(nf_0)\right)}$$

4-8 Ideal Filters (مرشح)

Filters are devices that is used in many applications to pass and block Some elements such as

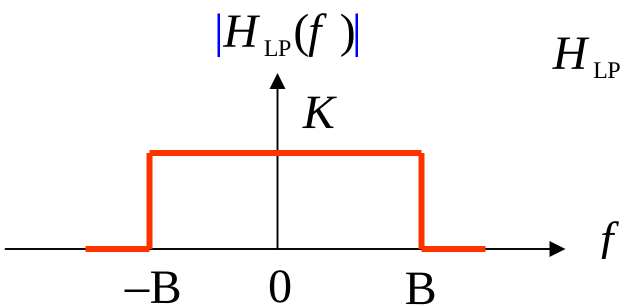
Water Filter , Air Filter and others

In signals, filters are used to pass some frequencies of the signal and Block other frequencies

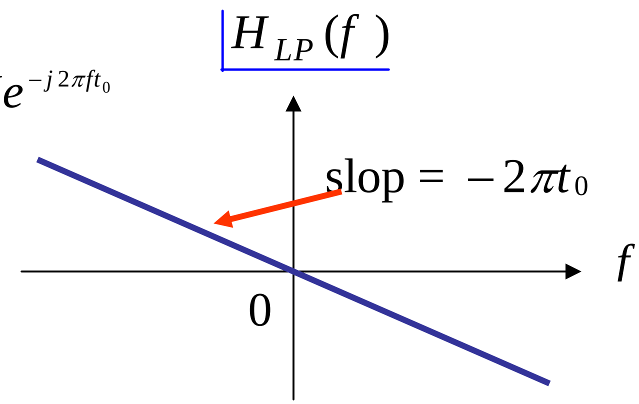
We are going to use the ideal filters

Three types of ideal filters that we will consider

Ideal Low pass Filter



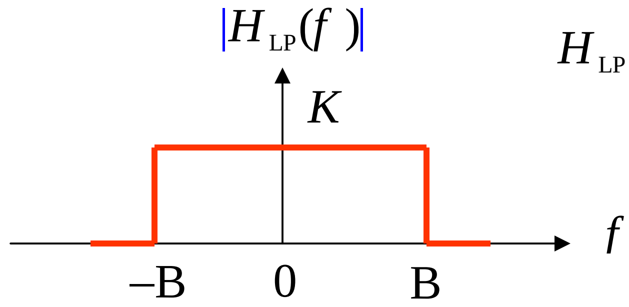
$$H_{LP}(f) = Ke^{-j2\pi f t_0}$$

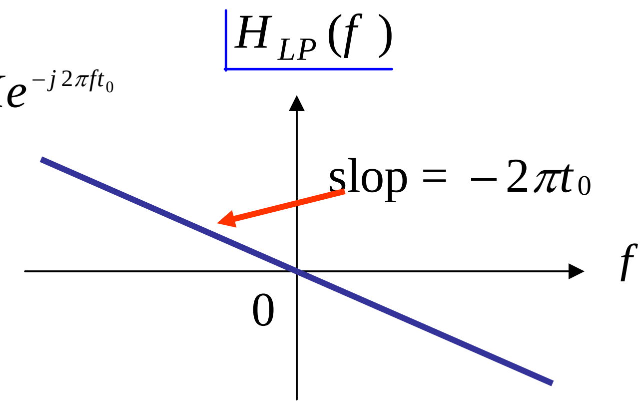


$$h_{LP}(t) = \int_{-\infty}^{\infty} H_{LP}(f) e^{j2\pi f t} df \quad = \int_{-B}^{B} (Ke^{-j2\pi f t_0}) e^{j2\pi f t} df$$

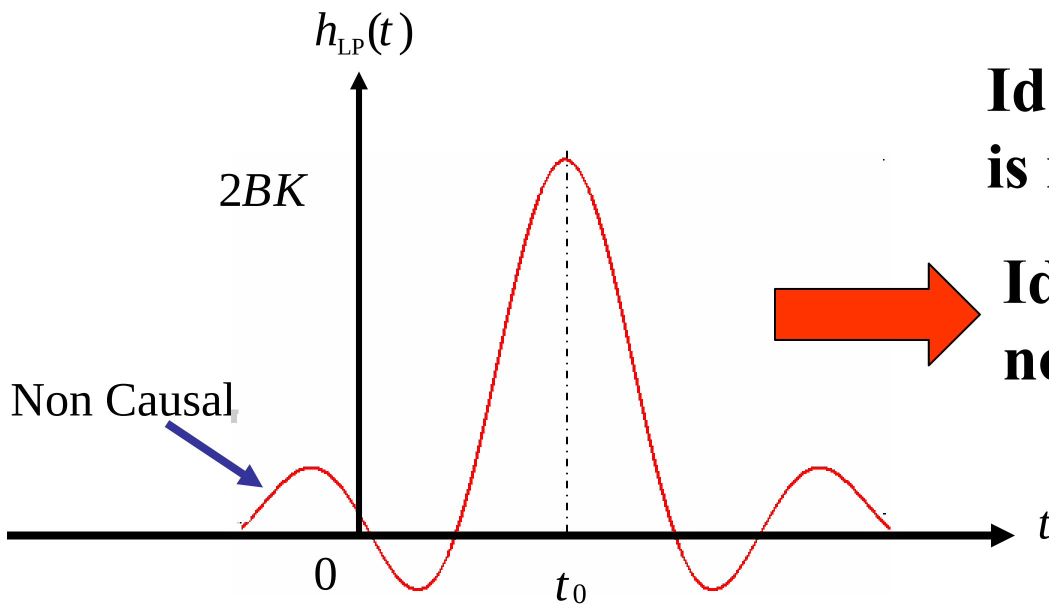
$$= \int_{-B}^{B} Ke^{j2\pi f (t - t_0)} df \quad = 2BK \operatorname{sinc}(2B(t - t_0))$$

$$H_{LP}(f) = Ke^{-j2\pi f t_0}$$





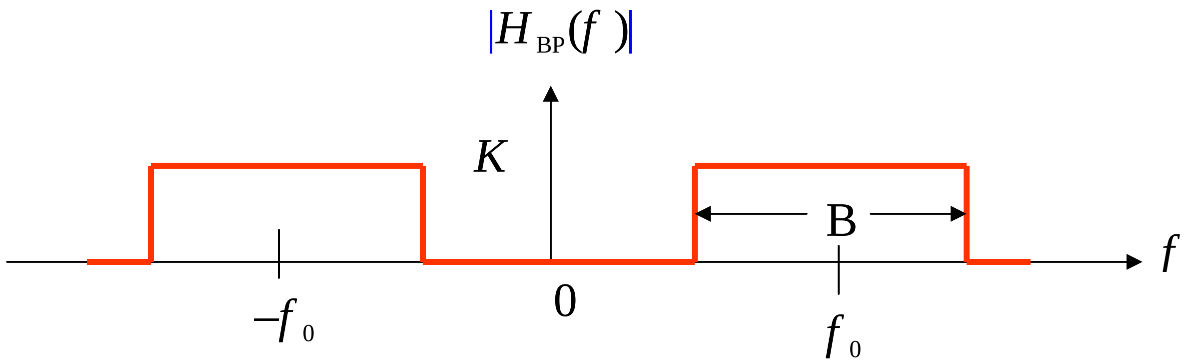
$$h_{LP}(t) = \int_{-\infty}^{\infty} H_{LP}(f)e^{j2\pi ft}df = \int_{-B}^{B}(Ke^{j2\pi ft_0})e^{j2\pi ft}df = 2BK\,\mathrm{sinc}(2B(t - t_0))$$

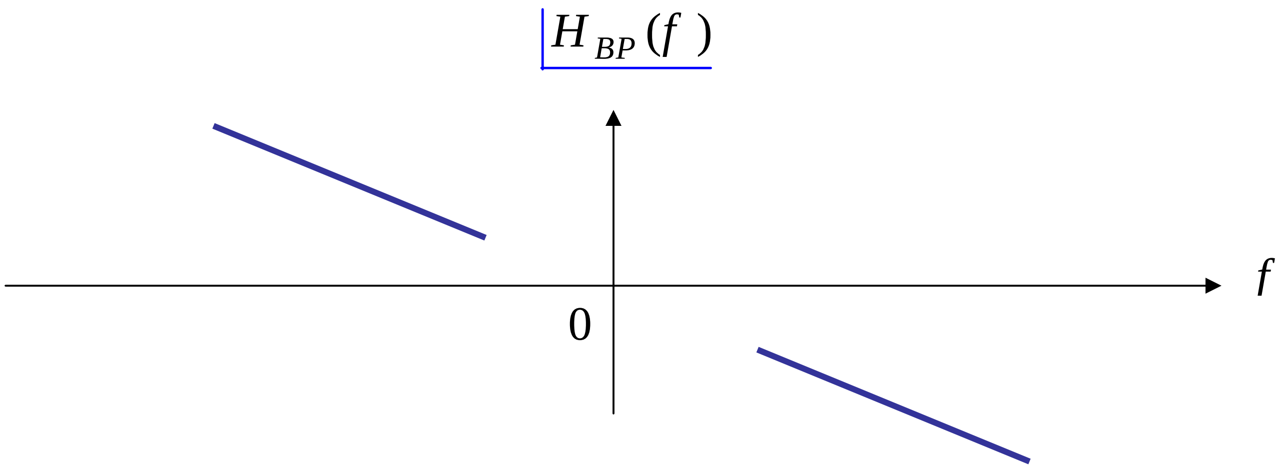


Ideal Low pass Filter is non causal

Ideal Low pass Filter non realizable

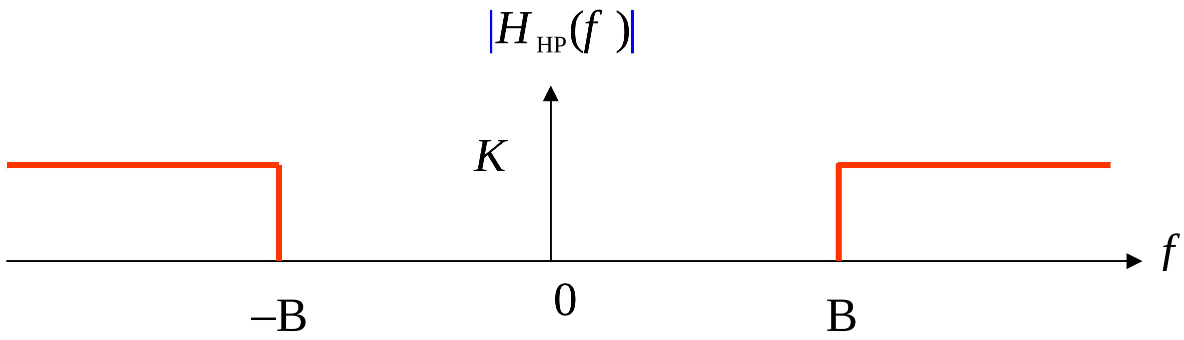
Ideal Band pass Filter

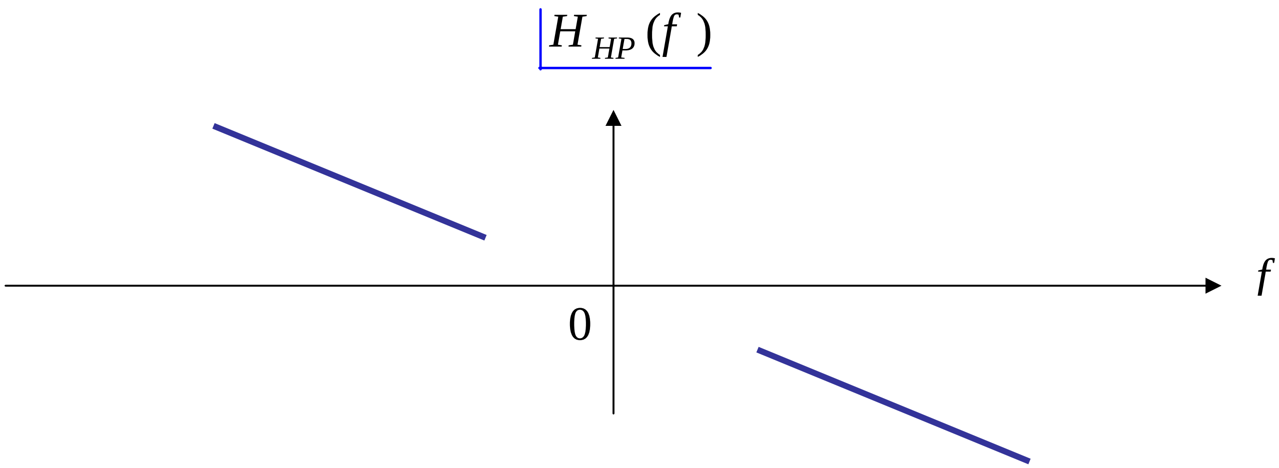




$$h_{BP}(t) = 2BK \text{sinc}(2B(t - t_0)) \cos[2\pi f_0 (t - t_0)]$$

Ideal High pass Filter





$$h_{HP}(t) = K\,\delta(t - t_0) - 2BK\,\mathrm{sinc}(2B(t - t_0))$$