End-to-End Outage Probability of Multihop Relayed Transmissions over Lognormal Shadowed Channels*

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***Abstract*— End-to-end outage probability evaluation of multihop wireless communication systems with non-regenerative relays over lognormal shadowing channels is presented. Closed-form expression for the end-to-end signal to noise ratio is derived. It is shown that this expression is in a form that makes it suitable to be approximated by a lognormal variate. Consequently, closed-form expressions for outage probability of non-regenerative systems can be readily obtained and then compared to that of regenerative systems. Numerical examples show that regeneration is more crucial at low average SNR and for multihop systems with a large number of hops.**

I. Introduction

Multihop transmission is a promising technique to achieve broader coverage and to mitigate wireless channels impairments. The main idea is that communication is achieved by relaying the information from the source to the destination via many intermediate terminals in between. The dual-hop transmission special case was encountered originally in bent-pipe satellites where the primary function of the spacecraft is to relay the uplink carrier into a downlink [1]. It is also common in various fixed microwave links by enabling greater coverage without the need of large power at the transmitter. More recently, this concept has gained new actuality in collaborative/cooperative wireless communication systems [2], [3], [4], [5], [6]. In this case, the key idea is that a mobile terminal relays a signal between the base station and a nearby mobile terminal when the direct link between the base station and the original mobile terminal is in deep fade. More generally, multihop transmission is common in multihop-augmented networks in which packets propagate through many hops before reaching their destination (see [7] and references therein). The performance of multihop transmissions for different fading channels can be found in [8] and [9].

In this paper, we focus on these multihop communication systems and study their end-to-end performance over independent, not necessarily identically distributed, lognormal shadowed channels. In addition to its proven empirical fit specially for low mobility terminals shadowed by large terrain and moving human bodies, surprisingly enough, the lognormal distribution has many other appealing characteristics from an analytical point of view in the context of this paper. These include: (i) The reciprocal of a lognormal variate is another lognormal variate, (ii) The product of two lognormal variates is a lognormal variate, and (iii) The sum of (uncorrelated or correlated) lognormal variates can be well approximated by another lognormal [10, Section 3.1, p. 129], using the Fenton-Wilkinson [10, Section 3.1.1, p. 130] or the Schwartz-Yeh methods [10, Section 3.1.2, p. 132] for the uncorrelated case or the recent extension of these methods by Pratesi *et al.* in [11] for the correlated case. The performance criterion used in this paper is outage probability which is the probability that the link quality falls below a predetermined threshold. Consequently, outage analysis captures the quality of performance that is guaranteed for a certain level of reliability.

The remainder of this paper is organized as follows. Next section introduces the system and channel models under consideration. Section III gives an analytical approach to evaluate the outage probability of non-regenerative systems as well as regenerative systems, and finally, section IV presents some numerical examples.

II. System and Channel Models

Consider the wireless communication system shown in Fig. 1. Here, signals propagate through N channels/hops before arriving to its destination. Intermediate terminals relay the signal from one hop to the next. For non-regenerative systems, these intermediate terminals amplify and forward the received signal from the previous node without any sort of decoding. This is referred to sometimes as analog relaying [7], in contrast to digital relaying that is employed in regenerative systems and which consists in fully decoding the received signal and then forwarding it to the next hop. In order to satisfy the average power constraint of the nth relay ($n = 1, \cdots, N-1$), its gain G_n is set to [3]

$$G_n^2 = \frac{1}{\alpha_n^2 + N_{0,n}}, \tag{1}$$

where α_n is the fading amplitude of the nth hop and $N_{0,n}$ is the power of the additive white Gaussian noise at the input of the

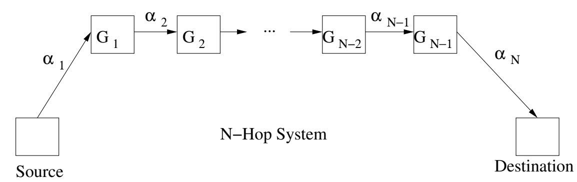

Fig. 1. Multihop communication system model.

*This work was supported in part by the Ministry of Education at the State of Qatar and in part by the National Science Foundation Grant CCR-9983462.

nth relay. The choice of the relay gain in (1) aims to invert the preceding channel effect while limiting the relay power if the fading amplitude of the inverted channel, α_n, is low. Under this relay gain set-up, we show in Appendix A that the end-to-end signal-to-noise ratio (SNR) is given by

$$\gamma_{\mathrm{eq}} = \left[\prod_{n=1}^{N} \left(1 + \frac{1}{\gamma_n}\right) - 1 \right]^{-1}, \qquad (2)$$

where $\gamma_n = \alpha_n^2 / N_{0,n}$ is the SNR of the nth hop. The N-hop end-to-end SNR expression in (2) is a generalization for the one given in [3, Eq. (21)] for the dual-hop end-to-end SNR. Under the lognormal shadowing assumption, $\gamma_n, n = 1, \cdots, N$, follow the lognormal distribution given by

$$p_{\gamma_n}(\gamma_n) = \frac{\zeta}{\sqrt{2\pi}\sigma_n \gamma_n} \exp\left[-\frac{(10\log_{10}\gamma_n - \mu_n)^2}{2\sigma_n^2} \right], \qquad (3)$$

where $\zeta = 10/\ln 10 = 4.3429$, μ_n (in dB) is the mean of $10\log_{10}\gamma_n$ and σ_n (in dB) is the standard deviation of $10\log_{10}\gamma_n$. Consequently, it can be shown that the equivalent SNR in (2) can be well approximated by a lognormal distribution. First, the term $1/\gamma_n$ is lognormally distributed following property (i). Next, $1 + 1/\gamma_n$ is well approximated by a lognormal distribution as per (iii). This is accomplished by dealing with the 1 as a lognormal distribution with zero mean and variance. The product of the resulting N lognormal variates is another lognormal variate as per (ii). Now, we are left with a lognormal variate with a -1 in front. We make the assumption that the difference of two lognormal variates is another lognormal variate, and we derive in Appendix B the modified Wilkinson equations for this case (note that we dealt again with 1 as a lognormal variate with zero mean and variance). Finally, the reciprocal of the resulting lognormal variate, which is also a lognormal variate, is the equivalent SNR.

III. Outage Probability

A. Non-Regenerative Systems

In noise limited non-regenerative systems, outage probability is defined as the probability that the instantaneous end-to-end equivalent SNR, γ_{eq}, falls below a predetermined protection ratio, γ_{th}, namely

$$P_{\mathrm{out}} = \mathrm{P}\left[\gamma_{\mathrm{eq}} \leq \gamma_{\mathrm{th}}\right] = \int_0^{\gamma_{\mathrm{th}}} p_{\Gamma_{\mathrm{eq}}}(\gamma)\, d\gamma = P_{\Gamma}(\gamma_{\mathrm{th}}). \qquad (4)$$

In (4), the predetermined protection ratio γ_{th} is a threshold SNR above which the quality of service is satisfactory and which essentially depends on the type of modulation employed and the type of application supported. Consequently, outage probability is given by

$$P_{\mathrm{out}} = Q\left(\frac{\mu_{\mathrm{eq}} - \gamma_{\mathrm{th}}}{\sigma_{\mathrm{eq}}}\right), \qquad (5)$$

where μ_{eq} and σ_{eq} are the mean and standard deviation of γ_{eq}, respectively. Another, yet simpler, way to evaluate outage probability proceeds as follows

$$\begin{aligned}
P_{\mathrm{out}} &= P[\gamma_{\mathrm{eq}} < \gamma_{\mathrm{th}}] \\
&= P\left[\left[\prod_{n=1}^{N} \left(1 + \frac{1}{\gamma_n}\right) - 1 \right]^{-1} < \gamma_{\mathrm{th}} \right] \\
&= P\left[\left[\prod_{n=1}^{N} \left(1 + \frac{1}{\gamma_n}\right) - 1 \right] > \frac{1}{\gamma_{\mathrm{th}}} \right] \\
&= P\left[\left[\prod_{n=1}^{N} \left(1 + \frac{1}{\gamma_n}\right) \right] > \frac{1}{\gamma_{\mathrm{th}}} + 1 \right] \\
&= P[\tilde{\gamma} > \frac{1}{\gamma_{\mathrm{th}}} + 1], \qquad (6)
\end{aligned}$$

where $\tilde{\gamma} = \prod_{n=1}^{N} \left(1 + \frac{1}{\gamma_n}\right)$. Consequently, outage probability is given alternatively by

$$P_{\mathrm{out}} = 1 - Q\left(\frac{\mu_{\tilde{\gamma}} - \frac{1}{\gamma_{\mathrm{th}}} - 1}{\sigma_{\tilde{\gamma}}}\right), \qquad (7)$$

where $\mu_{\tilde{\gamma}}$ and $\sigma_{\tilde{\gamma}}$ are the mean and standard deviation of $\tilde{\gamma}$ respectively. Note that evaluating outage probability as per (7) bypasses the need for the new approximation of the difference of two lognormal variates. However, these results are kept here for possible other applications. It was reported in the literature [12] that for outage probability calculations, the best method for approximating the sum of lognormal variates is the Fenton-Wilkinson method. Consequently, this method is used for the numerical examples presented in this paper.

B. Regenerative Systems

In regenerative systems, on the other hand, outage decisions are taken on a per hop basis, and the overall system outage is dominated by the weakest hop/link. Consequently, the outage probability is given by

$$P_{\mathrm{out}} = P[\mathrm{Min}(\gamma_1, \cdots, \gamma_N) < \gamma_{\mathrm{th}}]. \qquad (8)$$

In this case, outage probability can be shown to be given by

$$P_{\mathrm{out}} = 1 - \prod_{n=1}^{N} Q\left(\frac{\gamma_{\mathrm{th}} - \mu_n}{\sigma_n}\right), \qquad (9)$$

where μ_n and σ_n are the mean and standard deviation of the nth hop, respectively.

IV. Numerical Examples

Fig. 2 shows the outage probability performance of a two hops system as a function of $\mu - \gamma_{\mathrm{th}}$, where it is assumed that the two links are identically distributed. Two systems are compared here, namely regenerative and non-regenerative systems. As shown in the figure, regenerative systems outperform non-regenerative systems for low ranges of average SNR. The two systems perform the same for large average SNR. It is clear also that the Fenton-Wilkinson method closely matches the Monte Carlo simulation for most ranges of SNR. However, at high values of $\mu - \gamma_{\mathrm{th}}$, it starts deviating from the Monte Carlo simulation. This behavior is not noticed if the links have higher σ's as shown in Fig. 3, where a better match between the approximation and the Monte Carlo simulation is noticed. Fig. 4

studies the system outage probability as a function of the total number of identically distributed hops. Note that a diminishing increase in outage probability results from increasing the number of hops. Finally, Fig. 5 compares the performance of non-regenerative systems with that of regenerative systems as a function of the number of hops. It is clear that there is an increasing gap in the performance between the two systems as the number of hops increases which indicates that regeneration is more crucial if the number of hops is large.

Appendix A
Derivation of Eq. (2)

Consider the multihop communication system shown in Fig. 1. In order to get the SNR at the end of the Nth hop, we need to calculate the signal power and the noise power components at the destination hop, and then divide them to get the required SNR. These powers are given by

$$\begin{aligned}\text{Signal Power} &= (\alpha_1^2\alpha_2^2\cdots\alpha_N^2)(G_1^2G_2^2\cdots G_{N-1}^2)=\mathcal{S}_N.\\ \text{Noise Power} &= N_{0,1}(G_1^2G_2^2\cdots G_{N-1}^2)(\alpha_2^2\alpha_3^2\cdots\alpha_N^2)\\ &+ N_{0,2}(G_2^2G_3^2\cdots G_{N-1}^2)(\alpha_3^2\alpha_4^2\cdots\alpha_N^2)\\ &+ \cdots + N_{0,N}=\mathcal{N}_N. \end{aligned} \tag{10}$$

Consequently,

$$\begin{aligned}\gamma_{\mathrm{eq}_N} &= \frac{\mathcal{S}_N}{\mathcal{N}_N}\\ &= \frac{\prod_{n=1}^N \alpha_n^2 \prod_{n=1}^{N-1} G_n^2}{\sum_{n=1}^N N_{0,n}\prod_{t=n+1}^N \alpha_t^2 \prod_{t=n}^{N-1} G_t^2}. \end{aligned} \tag{11}$$

Dividing both the numerator and the denominator by $\prod_{n=1}^N N_{0,n}\prod_{n=1}^{N-1} G_n^2$, the numerator is now given by

$$\text{Numerator} = \prod_{n=1}^N \gamma_n, \tag{12}$$

where $\gamma_n=\alpha_n^2/N_{0,n}$ is the per hop SNR. The denominator is given by

$$\begin{aligned}\text{Denominator} &= \frac{\sum_{n=1}^N N_{0,n}\prod_{t=n+1}^N \alpha_t^2 \prod_{t=n}^{N-1} G_t^2}{\prod_{n=1}^N N_{0,n}\prod_{n=1}^{N-1} G_n^2}\\ &= \sum_{n=1}^N N_{0,n}\frac{\prod_{t=n+1}^N \alpha_t^2}{\prod_{t=1}^{n-1} G_t^2 \prod_{t=1}^N N_{0,t}}\\ &= \sum_{n=1}^N N_{0,n}\frac{\prod_{t=n+1}^N \gamma_t}{\prod_{t=1}^{n-1} G_t^2 \prod_{t=1}^n N_{0,t}}\\ &= \sum_{n=1}^N \frac{\prod_{t=n+1}^N \gamma_t}{\prod_{t=1}^{n-1} G_t^2 \prod_{t=1}^{n-1} N_{0,t}}. \end{aligned} \tag{13}$$

Using $G_t^2=\frac{1}{\alpha_t^2+N_{0,t}}$, the denominator becomes

$$\begin{aligned}\text{Denominator} &= \sum_{n=1}^N \frac{\prod_{t=n+1}^N \gamma_t}{\prod_{t=1}^{n-1}\frac{N_{0,t}}{\alpha_t^2+N_{0,t}}}\\ &= \sum_{n=1}^N \prod_{t=n+1}^N \gamma_t \prod_{t=1}^{n-1}(\gamma_t+1). \end{aligned} \tag{14}$$

The equivalent SNR is then given by

$$\begin{aligned}\gamma_{\mathrm{eq}_N} &= \frac{\prod_{n=1}^N \gamma_n}{\sum_{n=1}^N \prod_{t=n+1}^N \gamma_t \prod_{t=1}^{n-1}(\gamma_t+1)}\\ &= \left[\sum_{n=1}^N \frac{\prod_{t=1}^{n-1}(\gamma_t+1)}{\prod_{t=1}^n \gamma_t}\right]^{-1}\\ &= \left[\sum_{n=1}^N \frac{1}{\gamma_n}\prod_{t=1}^{n-1}\left(1+\frac{1}{\gamma_t}\right)\right]^{-1}, \end{aligned} \tag{15}$$

which upon expansion can be rewritten by inspection in the slightly simpler form as

$$\gamma_{\mathrm{eq}_N} = \left[\prod_{n=1}^N\left(1+\frac{1}{\gamma_n}\right)-1\right]^{-1}, \tag{16}$$

which is the required results.

Appendix B
Approximating the Difference of Two Lognormal Variates Using Fenton-Wilkinson Method

In this appendix, we will follow the steps of the summary of Fenton-Wilkinson method presented in [13]. Let I_1 and I_2 be two independent lognormal variates, i.e.

$$X_i = 10\log_{10} I_i = m_{X_i}+\chi_i, \tag{17}$$

where m_{X_i} is the area mean power in dBm, and χ_i is a zero-mean normally distributed random variable in dB with standard deviation σ_{X_i}, also in dB. Let $Y_i=\ln I_i=\zeta X_i$, where $\zeta=\ln 10/10$, then we assume that

$$I = I_1 - I_2 = e^{Y_1}-e^{Y_2}\approx e^Z = 10^X, \tag{18}$$

where Z (in logarithmic units) and X (in dB) are normally distributed, and $Z=\zeta X$. According to Wilkinson's method, the mean and standard deviation of Z are found by matching the first and second moments of I with that of I_1-I_2. Hence,

$$E\{e^Z\} = E\{e^{Y_1}-e^{Y_2}\} \tag{19}$$

$$E\{e^{2Z}\} = E\{(e^{Y_1}-e^{Y_2})^2\}. \tag{20}$$

Using the formula

$$E\{e^{nu}\} = \exp\left(nm_u+\frac{1}{2}n^2\sigma_u^2\right), \tag{21}$$

where u is a normal RV with mean m_u and variance σ_u^2, the moments in (19) and (20) can be written as

$$\begin{aligned}\exp\left(m_Z+\frac{\sigma_Z^2}{2}\right) &= \exp\left(m_{Y_1}+\frac{\sigma_{Y_1}^2}{2}\right)-\exp\left(m_{Y_2}+\frac{\sigma_{Y_2}^2}{2}\right)\\ &= u_1, \end{aligned} \tag{22}$$

and

$$\begin{aligned}\exp(2m_Z+2\sigma_Z^2) &= \exp(2m_{Y_1}+2\sigma_{Y_1}^2)+\exp(2m_{Y_2}+2\sigma_{Y_2}^2)\\ &- 2\exp(m_{Y_1}+m_{Y_2})\exp\left(\frac{\sigma_{Y_1}^2+\sigma_{Y_2}^2}{2}\right)\\ &= u_2. \end{aligned} \tag{23}$$

Solving for m_Z and σ_Z, and using $Z = \zeta X$ we get

$$
\begin{aligned}
m_X &= (1/\zeta)(2\ln u_1 - \frac{1}{2}\ln u_2), \\
\sigma_X &= (1/\zeta)\sqrt{\ln u_2 - 2\ln u_1}.
\end{aligned} \tag{24}
$$

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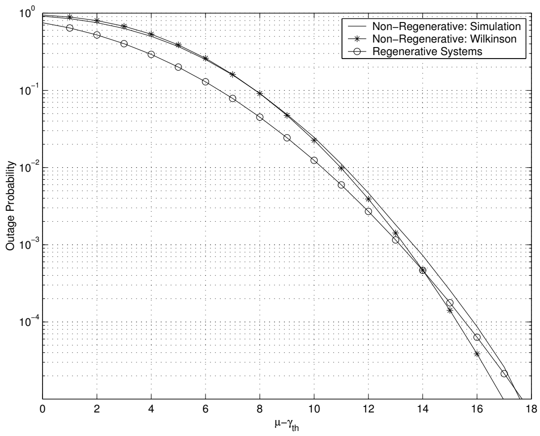

Fig. 2. Effect of changing $\mu - \gamma_{\rm th}$ on outage probability of a two hops system ($\sigma = 4$dB).

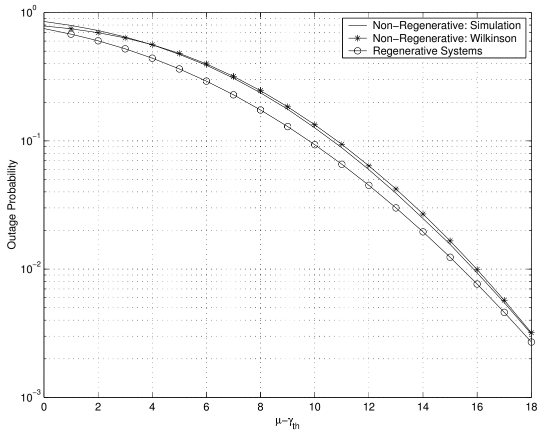

Fig. 3. Effect of changing $\mu - \gamma_{\rm th}$ on outage probability of a two hops system ($\sigma = 6$dB).

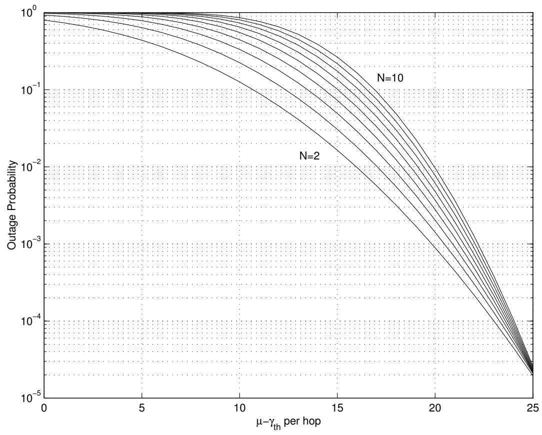

Fig. 4. Effect of increasing the number of hops on the performance of non-regenerative systems.

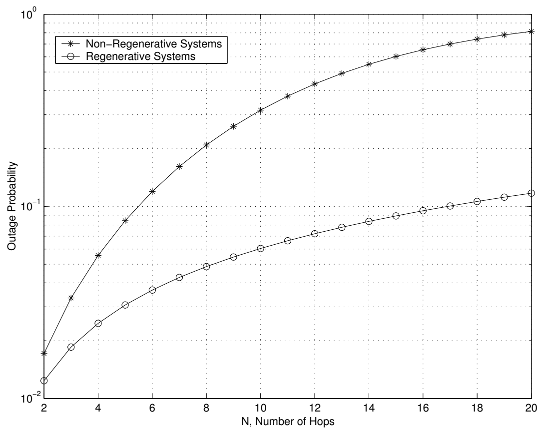

Fig. 5. Effect of increasing the number of hops on the end-to-end outage probability ($\mu - \gamma_{\rm th} = 15$dB per hop).