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Distance Properties of Space-Time Trellis Codes

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***Abstract*— Quasi-regularity is a code uniformity property that was defined for trellis codes to simplify performance evaluation. In the case of Space-time trellis codes over rapid fading channels, it was assumed, but not proven, that most codes in the literature are quasi-regular. In this paper, the definition of quasi-regularity of trellis codes is extended to ST trellis codes for rapid fading channels. This enables simplified performance evaluation using N-states only. Moreover, simulation and upper-bound examples of quasi-regular and non-quasi-regular codes are presented.**

I. Introduction

Space-time (ST) trellis codes (first introduced in their present form by Tarokh *et al* in [1]) rely on the idea of combining multiple transmit antennas with trellis-coded modulation. This resulted in a coding technique that provides both high transmission rate and strong error control capabilities for fading channels. Performance evaluation of ST trellis codes over rapid fading channels depends on their distance properties [1]. Simplifying performance evaluation requires that the code has some uniformity properties, such as quasi-regularity [2].

In this paper, the performance evaluation of ST trellis codes over rapid fading channels is presented. After presenting the quasi-regularity for trellis codes, we extend the definition to ST trellis codes. Finally, we demonstrate the performance of quasi-regular and non-quasi-regular codes.

II. System Model

A typical ST trellis encoder is shown in Fig 1, where a convolutional encoder takes k input bits and encodes them to $n \times \tilde{n}$ coded bits. Then these bits are divided into n sets, each consisting of $\tilde{n}$ bits. Every $\tilde{n}$ bits set is then mapped onto a point from a $2^{\tilde{n}}$-ary signal set. Finally, the resulting n symbols are modulated and transmitted at the same time instance and at the same frequency. The decoder consists of m receive antennas passing the received signals to m demodulators then to a maximal-ratio combiner and a Viterbi decoder.

In the following we present, briefly, the performance of ST trellis codes over a rapid fading channel. The received signal r_t^j at the j^{th} antenna at time t is a noisy superposition of all transmitted symbols over all transmit antennas and is given by:

$$r_t^j = \sum_{i=1}^{n} \alpha_{i,j}(t) c_t^i \sqrt{E_s} + \eta_t^j \tag{1}$$

where η_t^i is an AWGN modelled as independent samples of a zero-mean complex Gaussian random process with variance $N_0/2$ per dimension. The coefficient $\alpha_{i,j}(t)$ is the path gain from the i^{th} transmit antenna to the j^{th} receive antenna and c_t^i is the transmitted symbol from the i^{th} transmit antenna at time t.

At the receiver side, the Viterbi decoder computes a branch metric defined by the following:

$$\sum_{j=1}^{m} |r_t^j - \sum_{i=1}^{n} \alpha_{i,j}(t)\, c_t^i|^2 \tag{2}$$

Consider a rapid fading channel with independent fade coefficients (i.e. fade coefficients change independently, with time, from one fade coefficient to another). Define $\mathbf{C}_l$ as a codeword that has been transmitted over l time intervals and was erroneously decoded as $\hat{\mathbf{C}}_l$ where:

$$\mathbf{C}_l = c_1^1 c_1^2 \ldots c_1^n\, c_2^1 c_2^2 \ldots c_2^n c_l^1\, c_l^2 \ldots c_l^n$$
$$\hat{\mathbf{C}}_l = \hat{c}_1^1 \hat{c}_1^2 \ldots \hat{c}_1^n\, \hat{c}_2^1 \hat{c}_2^2 \ldots \hat{c}_2^n\, \hat{c}_l^1 \hat{c}_l^2 \ldots \hat{c}_l^n$$

Here, l is the length of the sequence in time and n is the number of transmit antennas, resulting in a total of $n \times l$ symbols in the codeword $\mathbf{C}_l$.

The pairwise error probability of deciding $\mathbf{C}_l$ in favor of $\hat{\mathbf{C}}_l$ for a rapid fading channel, assuming maximum likelihood decoding, is upper bounded by [1]:

$$P(\mathbf{C}_l, \hat{\mathbf{C}}_l) \leq \prod_{t \in \hat{\nu}} \left(1 + |\mathbf{c}_t - \hat{\mathbf{c}}_t|^2 \frac{E_s}{4N_0}\right)^{-m} \tag{3}$$

where $\hat{\nu}$ denote the set of time instances such that $|\mathbf{c}_t - \hat{\mathbf{c}}_t| \neq 0$, and

$$|\mathbf{c}_t - \hat{\mathbf{c}}_t|^2 = \sum_{i=1}^{n} |c_t^i - \hat{c}_t^i|^2 \tag{4}$$

It can be seen from equation (3) that in order to *minimize* the pairwise symbol error probability of space-time trellis codes over rapid fading channels, two main criteria must be *maximized* [1]:

- *The Distance Criterion:* The *symbol differences* between codewords $\mathbf{c}_t$ and $\hat{\mathbf{c}}_t$ should be maximized. When two codes have the same distance properties, the next criterion distinguishes them.
- *The Product Criterion:* The *minimum* of the following product should also be maximized:

$$\prod_{t \in \hat{\nu}} |\mathbf{c}_t - \hat{\mathbf{c}}_t|^2$$

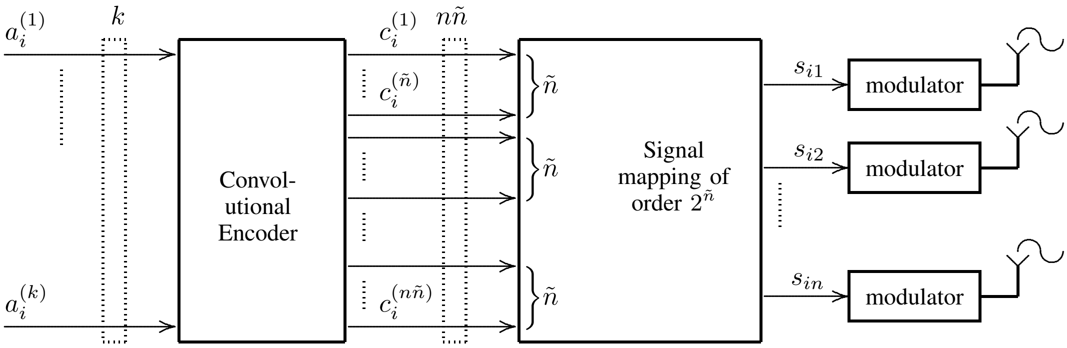

Fig. 1. General block diagram of a space-time trellis encoder

III. Distance Spectrum

One can investigate that there are many codes having the same minimum symbol-differences and minimum product-distance properties. Yet, these codes have different performance capabilities. For such codes, performance depends on higher *product distances* (not just the minimum) and the *multiplicities* of those paths (i.e. the number of paths with the same distance). Expanding the distance table to all distances and calculating their multiplicities yields the *distance spectrum*.

This can be seen in the bit error probability for a general trellis code, which is given by [3]:

$$P_b \leq \frac{1}{2k}\frac{\partial T(D,I)}{\partial I}\bigg|_{I=1} \tag{5}$$

where $T(D,I)$ is the transfer function of the error state diagram which enumerates all possible error sequences and is a function of the distance D and input bits I. The transfer function can be calculated from the distance spectrum.

Zummo and Al-Semari in [4] introduced a tightening constant to (5):

$$\begin{aligned} P_b \quad &\leq \quad \frac{1}{k}\left[\frac{1}{2^{2mL}}\sum_{j=1}^{mL}\binom{2mL-j-1}{mL-1}\left(\frac{2}{1+\sqrt{\phi_{\min}}}\right)^j\right] \\ &\times\frac{\partial T(D,I)}{\partial I}\bigg|_{I=1} \end{aligned} \tag{6}$$

where L is the cardinality of the minimum set of time instances in which the codewords $\mathbf{c}_t = c_t^1 c_t^2 \ldots c_t^n$ are different from the all-zero codeword and

$$\phi_{\min} = \min\left\{\frac{E_s|\mathbf{c}_t - \hat{\mathbf{c}}_t|^2}{4N_0 + E_s|\mathbf{c}_t - \hat{\mathbf{c}}_t|^2}, \qquad t \in \hat{\nu}\right\}$$

In order to calculate the distance spectrum of a general trellis code, one must calculate the product distance between all possible paths between any two states. Biglieri [5] described a general method that uses an expanded set of N^2 states. Needless to say that this method is very difficult for codes with relatively large number of states. Many researchers explored symmetries of certain trellis codes to reduce the number of states needed to calculate the distance spectrum without loss of generality (see for example [2], [6], [7] and [8]). In all these papers, it has been proven that the code symmetries enable to calculate the product distance between all possible paths and the all-zero path only [9].

IV. Quasi-Regularity of ST Trellis Codes

One such symmetry is quasi-regularity introduced by Rouanne and Costello [2]. In this approach, the Euclidian distances between output symbols are found as polynomials of error vector labels. If a code is quasi-regular, then its distance spectrum can be calculated assuming the all-zero path being transmitted.

Let σ and $\hat{\sigma}$ be two states in the code trellis (the notations here follow those of Benedetto *et al* [9]), and $\mathbf{e} = e_0, \ldots, e_k$ be a vector representing a binary $(k+1)$-tuple, called the *error vector*. The distance polynomial $P_{\sigma,\hat{\sigma},\mathbf{e}}(D)$ is defined as:

$$P_{\sigma,\hat{\sigma},\mathbf{e}}(D) = \sum_{\mathbf{a}|\sigma} p(\mathbf{a}|\sigma) D^{\mathrm{d}^2[s(\mathbf{a}),\, s(\mathbf{a}\oplus\mathbf{e})]} \tag{7}$$

where $p(\mathbf{a}|\sigma)$ is the probability that the binary label vector $\mathbf{a}$ labels a transition originating from the state σ, and $d^2[s(\mathbf{a}), s(\mathbf{a}\oplus\mathbf{e})]$ is the squared Euclidean distance between signals $s(\mathbf{a})$ (originating from σ) and signals $s(\mathbf{a}\oplus\mathbf{e})$ (originating from $\hat{\sigma}$), which are labelled by two binary $(k+1)$-tuples which differ by $\mathbf{e}$. The summation in (7) is extended over all vectors $\mathbf{a}$ labelling transitions originating from σ.

A trellis scheme is said to be quasi-regular if [2]:

1) its encoder is linear,
2) for all $\mathbf{e}$ and for all pairs of states $(\sigma_i, \hat{\sigma_i})$ and $(\sigma_j, \hat{\sigma_j})$

$$P_{\sigma_i,\hat{\sigma_i},\mathbf{e}}(D) = P_{\sigma_j,\hat{\sigma_j},\mathbf{e}}(D) \tag{8}$$

To verify condition (2), each pair of states must be considered and the polynomials for each admissible error vector $\mathbf{e}$ must be constructed.

In the case of space-time trellis codes, it should be noted that unlike trellis codes, ST trellis codes have more than one output symbol. Therefore, the labels for the error vectors and for the output symbols are used n-times (where n is the number of output antennas). To demonstrate this consider the code of Fig 2. For the output symbols (2 3), the labels are (10 11), where the (10) corresponds to the output symbol of the first

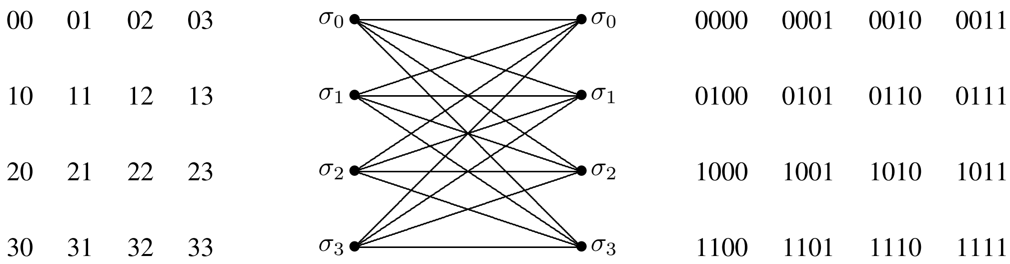

Fig. 2. Trellis diagram of Tarokh Code [1] and the binary labelling vectors

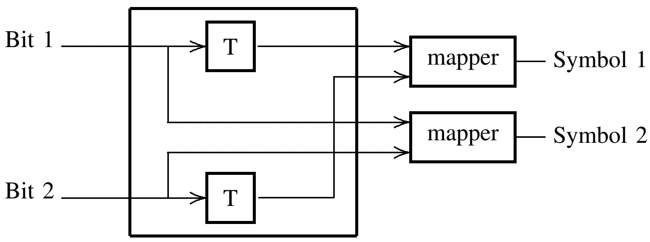

Fig. 3. Encoder of the code in Fig 2

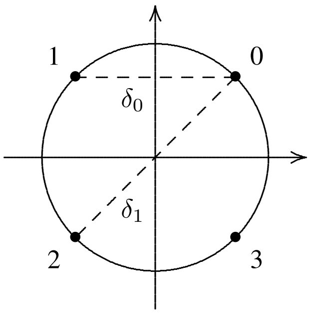

Fig. 4. Constellation points of a QPSK system

antenna (2) and (11) corresponds to the output symbol of the second antenna (3).

Example 1: To demonstrate quasi-regularity of ST trellis codes, we check quasi regularity of the code intruduced by Tarokh *et al* [1] shown in Fig 2.

This code can be realized as a linear encoder followed by a mapper as can be seen from Fig 3. The next step is to verify the second condition of quasi-regularity definition.

Consider the two states σ_0 and σ_1 and the error vector $\mathbf{e} = \{0000\}$. Because the binary labels of state σ_0 (0000,0001,0010,0011) are different from those originating from state σ_1 (0100,0101,0110,0111), the polynomial $P_{0,1,0000}$ is not defined and we can proceed to the next error vector $\mathbf{e} = \{0100\}$. In this case the polynomial is defined because the set of labels of the signals leaving σ_1 is obtained by adding the error vector $\mathbf{e} = \{0100\}$ to the set of labels leaving state σ_0. Therefore the polynomial $P_{0,1,0100}$ is given by:

$$\begin{aligned} P_{0,1,0100}(D) &= p(0000|0)D^{\mathrm{d}^2[s(0000),s(0100)]} + \ldots \\ &\quad + p(0011|0)D^{\mathrm{d}^2[s(0011),s(0111)]} \\ &= D^{\delta_0^2} \end{aligned}$$

TABLE I
EUCLIDEAN DISTANCES FOR ERROR VECTORS

		error vector			
symbol	label	00	01	10	11
0	00	0	δ_0^2	δ_1^2	δ_0^2
1	01	0	δ_0^2	δ_1^2	δ_0^2
2	10	0	δ_0^2	δ_1^2	δ_0^2
3	11	0	δ_0^2	δ_1^2	δ_0^2

where Euclidean distances δ_0 and δ_1 are as given in Fig 4.

Then we continue the search for other pairs of states with the error vector $\mathbf{e} = \{0100\}$ between its labels. We find that $P_{2,3,0100}$ is defined, and:

$$\begin{aligned} P_{2,3,0100}(D) &= p(0000|0)D^{\mathrm{d}^2[s(0000),s(0100)]} + \ldots \\ &\quad + p(0011|0)D^{\mathrm{d}^2[s(0011),s(0111)]} \\ &= D^{\delta_0^2} \end{aligned}$$

It can be seen that there are no other pairs of states where the error vector $\mathbf{e} = \{0100\}$ is defined, making $P_{0100} = D^{\delta_0^2}$. Doing the previous procedure for all possible error vectors that are defined and all pairs of states, the following distance polynomials are obtained:

$$\begin{array}{ll} P_{0000} = 1, & P_{0100} = D^{\delta_0^2}, \\ P_{1000} = D^{\delta_1^2}, & P_{1100} = D^{\delta_0^2}, \\ P_{0001} = D^{\delta_0^2}, & P_{0101} = D^{2\delta_0^2}, \\ P_{1001} = D^{\delta_0^2+\delta_1^2}, & P_{1101} = D^{2\delta_0^2}, \\ P_{0010} = D^{\delta_0^2}, & P_{0110} = D^{\delta_0^2+\delta_1^2}, \\ P_{1010} = D^{2\delta_1^2}, & P_{1110} = D^{\delta_0^2+\delta_1^2}, \\ P_{0011} = D^{\delta_0^2}, & P_{0111} = D^{2\delta_0^2}, \\ P_{1011} = D^{\delta_0^2+\delta_1^2}, & P_{1111} = D^{2\delta_0^2} \end{array}$$

One can check that for all $\mathbf{e}$ and for all pairs of states $(\sigma_i, \hat{\sigma_i})$ and $(\sigma_j, \hat{\sigma_j})$: $P_{\sigma_i,\hat{\sigma_i},\mathbf{e}}(D) = P_{\sigma_j,\hat{\sigma_j},\mathbf{e}}(D)$. The two conditions hold for this code, proving its quasi-regularity.

V. Examples of ST trellis Codes and Their Performance

In this section we provide examples of quasi-regular and non-quasi-regular ST trellis codes and their performance using both bounds presented previously (bound 1 from equation (5)

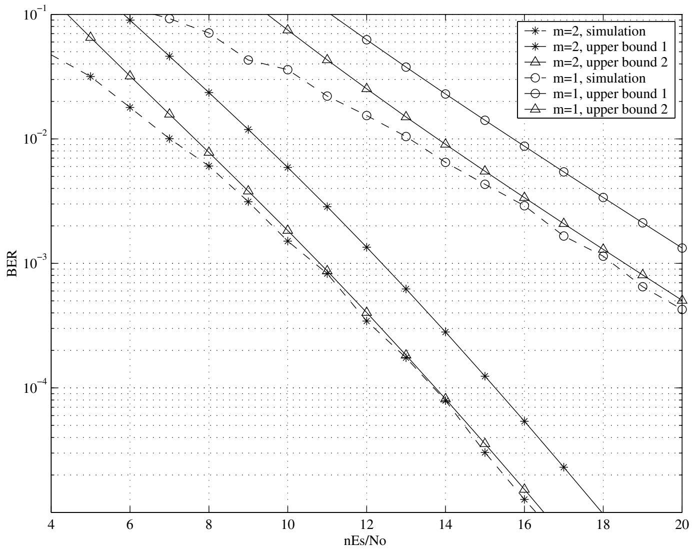

Fig. 5. Code of Ex. 2 – simulation and upper bounds for m=1 & 2

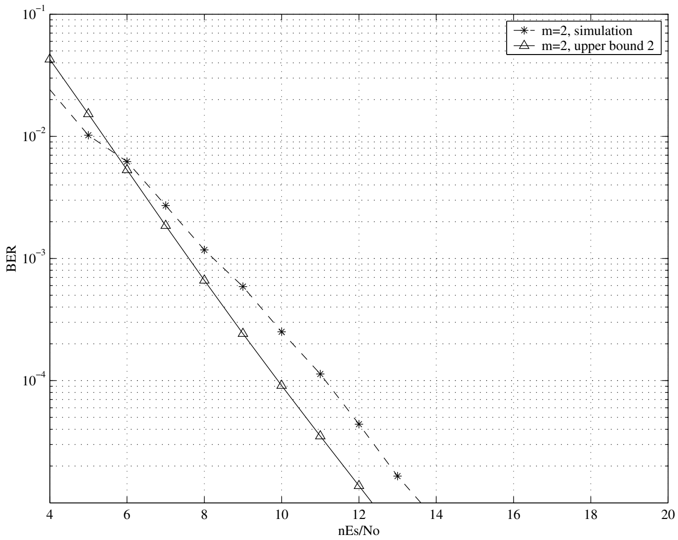

Fig. 7. Code of Ex. 3 – simulation and upper bound for m=1

00	21	32	12
23	02	10	31
33	20	01	13
22	03	11	30

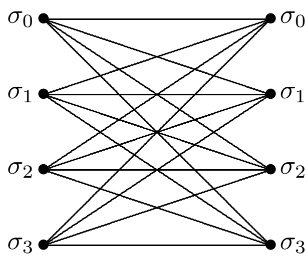

Fig. 6. Non-quasi-regular code of Ex. 3

and bound 2 from equation (6)) and using simulation over a rapid fading channel.

Example 2: Consider the code in Fig 2. It was demonstrated in example 1 that this code is quasi-regular. Therefore, bounds 1 & 2 can be applied assuming the all-zero codeword being transmitted without loss of generality. The quantity $\frac{\partial T(D,I)}{\partial I}\big|_{I=1}$ can be calculated from the expanded distance spectrum, and the distance D is evaluated for fading channels as [10]:

$$D^{\mathrm{d}^2[s(\mathbf{a}),\, s(\mathbf{a}\oplus\mathbf{e})]} = \left(1 + \sum_{i=1}^{n} |c_t^i - \hat{c}_t^i|^2 \frac{E_s}{4N_0}\right)^{-m} \quad (9)$$

The results are shown in Fig 5. It is clear from simulation results that the code is upper bounded by both bounds for the cases of one and two receive antennas.

Example 3: Consider the code in Fig 6. Investigating the first condition of quasi-regularity, we find that this code cannot be realized using a linear encoder, because it requires *AND* gates. Furthermore, this code does not abide with the second condition of quasi-regularity. For example $P_{2,3,0100}$ is defined for some but not all signals between the two states σ_2 and σ_3.

The simulation and upper bounds calculations are shown in fig 7, which shows that this code is not upper bounded by the tighter bound of equation 6 and hence showing its non-quasi-regularity.

VI. Summary and Conclusions

In this paper, we extended the definition of quasi-regularity to space-time trellis codes. Examples of quasi-regular and non-quasi-regular space-time trellis codes were demonstrated. Moreover, the performance of these codes based on the distance properties were evaluated and compared with simulation results.

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