COE 561, Term 111
Digital System Design and Synthesis
HW# 3 Solution

Due date: Saturday, Dec. 3

Q.1. Consider the following function:

$x = a\,c\,e + a'\,d' + a'\,e' + b\,c\,e + b'\,d' + b'\,e' + d\,e$

(i) Compute all the kernels of X using the recursive kernel computation algorithm. Show all the steps.

(ii) Compute all the kernels of X based on matrix representation. Compare your answer to the result obtained in (i).

(iii) Use the sis command ***print_kernel*** and compare the kernels obtained to your answers in (i) and (ii).

(iv) Find a good factor of X. Assume that input variables are sorted in lexicographic order. Determine the number of literals obtained. Compare your solution with the result obtained by running the sis commands ***factor –g x; print_factor; print_stats –f.***

Q.2. Consider the following function:

$x = a\,b\,d + a\,b'\,d' + a'\,c\,d + a'\,c'\,d' + e\,g\,h + e'\,f'\,g' + e'\,f'\,h' + e'\,f'\,i\,j + e'\,f'\,i' + e'\,f'\,j' + f\,g\,h$

(i) Compute all double-cube divisors of x along with their bases and their weights. Show only double-cube divisors that have non-empty bases.

(ii) Apply the fast extraction algorithm based on extracting double-cube divisors along with complements or single-cube divisors with two-literals. Show all steps of the algorithm. Determine the number of literals saved. Compare your solution with the result obtained by running the sis commands ***fx***.

Q.3. Consider the logic network defined by the following expressions:

$X = A\,B;$
$Y = A\,B\,C\,X + A\,B'\,C'\,X';$
$Z = A' + Y;$

Inputs are {A, B, C} and output is {Z}.

(i) Compute the CDC set for the cut at the inputs of circuit Y.

(ii) Compute the ODC set for node Y.

(iii) Simplify the function of Y using both its ODC and CDC.

(iv) Compute the ODC set for node X based on the resulting simplified function and simplify its function.

(v) Apply the sis command ***full_simplify*** and compare the solution obtained with your obtained solution based in (iii) & (iv).

Q.4. Consider the logic network defined by the following expressions:

$$f = a + b$$

$$g = f\,c$$

$$h = f\,d$$

$$i = g + h$$

$$j = i\,e$$

$$k = b'\,c'\,d'$$

$$x = j + k$$

Inputs are {a, b, c, d, e} and output is {x}. Assume that the delay of a gate is related to the number of its inputs. Also, assume that the input data-ready times are zero except for input a, which is equal to 2.

(i) Draw the logic network graph and compute the **data ready times** and **slacks** for all vertices in the network.

(ii) Determine the **maximum propagation delay** and the **topological critical path**.

(iii) Suggest an implementation of the function ***x*** to reduce the delay of the circuit. What is the **maximum propagation delay** after the modified implementation?

HW#3

Q1. $X = ace + \bar{a}\bar{d} + \bar{a}\bar{e} + bce + \bar{b}\bar{d} + \bar{b}\bar{e} + de$

(i) Recursive Kernel Computation

We assume that the variables are ordered in lexicographic order : $\{a, \bar{a}, b, \bar{b}, c, d, \bar{d}, e, \bar{e}\}$

$i=1$: $\{a\}$

Cubes containing a : $\{ace\} < 2 \Rightarrow$ no Kernels found

$i=2$: $\{\bar{a}\}$

Cubes containing $\bar{a}$: $\{\bar{a}\bar{d}, \bar{a}\bar{e}\} \geq 2$

$C = \bar{a}$

Kernel found: $\bar{d} + \bar{e}$

Recursive call on the Kernel with $i=3$ $\{b\}$ will not generate any additional Kernel since the number of cubes containing each variable is < 2.

$i=3$: $\{b\}$

Cubes containing b : $\{bce\} < 2 \Rightarrow$ no Kernels found

$i=4$: $\{\bar{b}\}$

Cubes containing $\bar{b}$: $\{\bar{b}\bar{d}, \bar{b}\bar{e}\} \geq 2$

$C = \bar{b}$

Kernel found : $\bar{d} + \bar{e}$

Recursive call on the Kernel with $i=5$ will not produce any Kernel.

$i = 5 : \{c\}$

Cubes containing c : $\{ace, bce\} \geq 2$

$C = ce$

Kernel found : $a + b$

Recursive call on the kernel with $i = 6$ will not produce any kernel.

$i = 6 : \{d\}$

Cubes containing d : $\{de\} < 2 \Rightarrow$ no kernels found

$i = 7 : \{\bar{d}\}$

Cubes containing $\bar{d}$: $\{\bar{a}\bar{d}, \bar{b}\bar{d}\} \geq 2$

$C = \bar{d}$

Kernel found : $\bar{a} + \bar{b}$

Recursive call on the kernel with $i = 8$ will not produce any kernel.

$i = 8 : \{e\}$

Cubes containing e : $\{ace, bce, de\}$

$C = e$

Kernel found : $ac + bc + d$

Recursive call on the kernel with $i = 9$ will not produce any kernel.

$i = 9 : \{\bar{e}\}$

Cubes containing $\bar{e}$: $\{\bar{a}\bar{e}, \bar{b}\bar{e}\} \geq 2$

$C = \bar{e}$

Kernel found : $\bar{a} + \bar{b}$

Recursive call on the kernel will not produce any kernel.

Thus, the set of Kernels and their Co-Kernels are

Kernel	Co-Kernel
$\bar{d}+\bar{e}$	$\bar{a}$, $\bar{b}$
$a+b$	ce
$\bar{a}+\bar{b}$	$\bar{d}$, $\bar{e}$
$ac+bc+d$	e

(ii) Kernel computation using matrix representation:

Cube	Var. R/C	a 1	$\bar{a}$ 2	b 3	$\bar{b}$ 4	c 5	d 6	$\bar{d}$ 7	e 8	$\bar{e}$ 9
ace	1	1				1			1	
$\bar{a}\bar{d}$	2		1					1		
$\bar{a}\bar{e}$	3		1							1
bce	4			1		1			1	
$\bar{b}\bar{d}$	5				1			1		
$\bar{b}\bar{e}$	6				1					1
de	7						1		1	

Prime Rectangle	Cube	Kernel
({2,3},{2})	$\bar{a}$	$\bar{d}+\bar{e}$
({5,6},{4})	$\bar{b}$	$\bar{d}+\bar{e}$
({1,4},{5,8})	ce	$a+b$
({2,5},{7})	$\bar{d}$	$\bar{a}+\bar{b}$
({3,6},{9})	$\bar{e}$	$\bar{a}+\bar{b}$
({1,4,7},{8})	e	$ac+bc+d$

The same set of Kernels are obtained as in (i)

(iii) Computing kernels using SIS:

```
sis> read_eqn hw3q1.eqn
sis> print
    {x} = a c e + a' d' + a' e' + b c e + b' d' + b' e' + d e
sis> print_kernel
Kernels of {x}
(a') * (d' + e')
(b') * (d' + e')
(c e) * (a + b)
(d') * (a' + b')
(e) * (a c + b c + d)
(e') * (a' + b')
(-1-) * (a c e + a' d' + a' e' + b c e + b' d' + b' e' + d e)
sis>
```

(iv) Good factoring

We need to compute the value of each kernel and select the kernel with the highest value.

The value of a kernel in factored form is different than the one used for extracting a kernel $= n\ell - \ell + (c-1)\sum_{i=1}^{n} |cK_i|$

Kernel	Co-kernel	value
$\bar{d} + \bar{e}$	$\bar{a}, \bar{b}$	$2\times 2 - 2 + (2-1)\times 2 = 4$
$a+b$	ce	$1\times 2 - 2 + (2-1)\times 2 = 2$
$\bar{a} + \bar{b}$	$\bar{d}, \bar{e}$	$2\times 2 - 2 + (2-1)\times 2 = 4$
$ac+bc+d$	e	$1\times 5 - 5 + (3-1)\times 1 = 2$

Thus, we can select either the divisor $\bar{d}+\bar{e}$ or $\bar{a}+\bar{b}$, Let us select $\bar{a}+\bar{b}$

$\Rightarrow \quad X = (\bar{a}+\bar{b})(\bar{d}+\bar{e}) + ace + bce + de$ 12 lit.

Factoring is then applied recursively on the divisor, quotient and remainder.
The divisor and quotient can't be factored further.
Then, we need to compute the kernels of the remainder $= ace + bce + de$

Kernel	Co-kernel	value
$a+b$	ce	$1\times 2 - 2 + (2-1)\times 2 = 2$
$ac+bc+d$	e	$1\times 5 - 5 + (3-1)\times 1 = 2$

Thus, any one can be selected. Let us select $ac+bc+d$

$\Rightarrow \quad X = (\bar{a}+\bar{b})(\bar{d}+\bar{e}) + (ac+bc+d)e$ 10 lit

The divisor $ac+bc+d$ has the kernel $a+b$ which has value $= 1\times 2 - 2 + (2-1)\times 1 = 1$

$\Rightarrow \quad X = (\bar{a}+\bar{b})(\bar{d}+\bar{e}) + ((a+b)c+d)e$ 9 lit.

<u>Good Factoring using SIS</u>:

```
sis> print
     {x} = a c e + a' d' + a' e' + b c e + b' d' + b' e' + d e
sis> factor -g x
sis> print_factor
    {x} = e (c (b + a) + d) + (b' + a') (e' + d')
sis> print_stats -f
hw3q1.eqn  pi= 5  po= 1  node=  1  latch= 0  lits(sop)=  16  lits(ff)
sis>
```

Q2 $x = abd + a\bar{b}\bar{d} + \bar{a}cd + \bar{a}\bar{c}\bar{d} + egh + \bar{e}f\bar{g}$
$+ \bar{e}f\bar{h} + \bar{e}fij + \bar{e}f\bar{i} + \bar{e}f\bar{j} + fgh$

(i) Double-Cube Divisors — 34 literals

Double-Cube	Base	Weight
$bd + \bar{b}\bar{d}$	a	$1\times4-1-4+1 = 0$
$ab + \bar{a}c$	d	$2\times4-2-4+2 = 4$
$a\bar{b} + \bar{a}\bar{c}$	$\bar{d}$	$2\times4-2-4+2 = 4$
$cd + \bar{c}\bar{d}$	$\bar{a}$	$1\times4-1-4+1 = 0$
$e + f$	gh	$1\times2-1-2+2+5 = 6$
$\bar{g} + \bar{h}$	$\bar{e}f$	$1\times2-1-2+2+2 = 3$
$\bar{g} + ij$	$\bar{e}f$	$1\times3-1-3+2 = 1$
$\bar{g} + \bar{i}$	$\bar{e}f$	$1\times2-1-2+2 = 1$
$\bar{g} + \bar{j}$	$\bar{e}f$	$1\times2-1-2+2 = 1$
$\bar{h} + ij$	$\bar{e}f$	$1\times3-1-3+2 = 1$
$\bar{h} + \bar{i}$	$\bar{e}f$	$1\times2-1-2+2 = 1$
$\bar{h} + \bar{j}$	$\bar{e}f$	$1\times2-1-2+2 = 1$
$ij + \bar{i}$	$\bar{e}f$	$1\times3-1-3+2 = 1$
$ij + \bar{j}$	$\bar{e}f$	$1\times3-1-3+2 = 1$
$\bar{i} + \bar{j}$	$\bar{e}f$	$1\times2-1-2+2+1 = 2$

(ii) Fast Extraction

From part (i), we can see that $W_{d\,max} = 5$ for the double-cube divisor $d = e+f$

The single-cube divisor with highest weight is $\bar{e}\bar{f}$ with $W_s = 4$

Thus, we extract the double-cube divisor $d = e+f$ and the resulting network is;

$$[1] = e+f$$

$$x = abd + a\bar{b}\bar{d} + \bar{a}cd + \bar{a}\bar{c}\bar{d} + [1]gh + \overline{[1]}\bar{g} + \overline{[1]}\bar{h} + \overline{[1]}ij + \overline{[1]}\bar{i} + \overline{[1]}\bar{j}$$

28 literals

Next, double-cube divisors are updated in the same way;

Double-Cube	Base	Weight
$bd + \bar{b}\bar{d}$	a	$1\times4-1-4+1=0$
$ab+\bar{a}c$	d	$2\times4-2-4+2=4$
$a\bar{b}+\bar{a}\bar{c}$	$\bar{d}$	$2\times4-2-4+2=4$
$cd+\bar{c}\bar{d}$	$\bar{a}$	$1\times4-1-4+1=0$
$\bar{g}+\bar{h}$	$\overline{[1]}$	$1\times2-1-2+1+1=1$
$\bar{g}+ij$	$\overline{[1]}$	$1\times3-1-3+1=0$
$\bar{g}+\bar{i}$	$\overline{[1]}$	$1\times2-1-2+1=0$
$\bar{g}+\bar{j}$	$\overline{[1]}$	$1\times2-1-2+1=0$
$\bar{h}+ij$	$\overline{[1]}$	$1\times3-1-3+1=0$
$\bar{h}+\bar{i}$	$\overline{[1]}$	$1\times2-1-2+1=0$
$\bar{h}+\bar{j}$	$\overline{[1]}$	$1\times2-1-2+1=0$
$ij+\bar{i}$	$\overline{[1]}$	$1\times3-1-3+1=0$

Double Cube	Base	Weight
$ij + \bar{j}$	$\overline{[1]}$	$1 \times 3 - 1 - 3 + 1 = 0$
$\bar{i} + \bar{j}$	$\overline{[1]}$	$1 \times 2 - 1 - 2 + 1 + 1 = 1$

Both double-cube divisors $ab + \bar{a}c$ and $a\bar{b} + \bar{a}\bar{c}$ have the highest weight of 4 and any one can be extracted. Let us extract $ab + \bar{a}c$ and the resulting network is:

$$[1] = e + f$$
$$[2] = ab + \bar{a}c$$
$$x = [2]d + \overline{[2]}\,\bar{d} + [1]gh + \overline{[1]}\,\bar{g} + \overline{[1]}\,\bar{h} + \overline{[1]}\,ij + \overline{[1]}\,\bar{i} + \overline{[1]}\,\bar{j}$$

24 literals

The process is repeated and either $\bar{g} + \bar{h}$ or $\bar{i} + \bar{j}$ can be extracted as they have the highest weight of 1. The resulting network is:

$$[1] = e + f$$
$$[2] = ab + \bar{a}c$$
$$[3] = \bar{g} + \bar{h}$$
$$x = [2]d + \overline{[2]}\,\bar{d} + [1]\overline{[3]} + [3]\overline{[1]} + \overline{[1]}\,ij + \overline{[1]}\,\bar{i} + \overline{[1]}\,\bar{j}$$

23 literals

Finally, the double cube $\bar{i} + \bar{j}$ is extracted as it has a weight of 1 resulting in:

$$[1] = e + f \qquad [2] = ab + \bar{a}c$$
$$[3] = \bar{g} + \bar{h} \qquad [4] = \bar{i} + \bar{j}$$
$$x = [2]d + \overline{[2]}\,\bar{d} + [1]\overline{[3]} + [3]\overline{[1]} + \overline{[1]}\,\overline{[4]} + \overline{[1]}\,[4]$$

22 literals

Fast Extraction using SIS:

```
sis> read_eqn hw3q2.eqn
sis> print
    {x} = a b d + a b' d' + a' c d + a' c' d' + e g h + e' f' g' + e' f' h' + e' f' i j + e' f' i' + e' f' j'+
f g h
sis> fx
sis> print
    {x} = [1] [3]' + [1]' [3] + [1]' [4] + [1]' [4]' + [2] d + [2]' d'
    [1] = e + f
    [2] = a b + a' c
    [3] = g' + h'
    [4] = i' + j'
sis> print_stats
hw3q2.eqn  pi=10  po= 1  node=  5  latch= 0  lits(sop)=  22  lits(ff)=  20
sis>
```

Q3 $X = AB$

$Y = ABCX + A\bar{B}\overline{CX}$

$Z = \bar{A} + Y$

(i) The cut is $\{A, B, C, X\}$

$SDC_X = x \oplus AB = \bar{x}AB + x\bar{A} + x\bar{B}$

$\Rightarrow CDC_{cut} = \bar{X}AB + X\bar{A} + X\bar{B}$

(ii) $ODC_y = \bar{A}$

(iii) Simplification of Y using CDC & ODC:

AB \ CX	00	01	11	10
00	X	X	X	X
01	X	X	X	X
11	X	0	1	X
10	1	X	X	0

$\Rightarrow Y = \bar{C}\bar{X} + CX$

(iv) $ODC_X = \bar{A}$

A \ B	0	1
0	X	X
1	0	1

$\Rightarrow X = B$

Thus, the resulting simplified network is:

$x = B$

$Y = \bar{C}\bar{X} + CX$

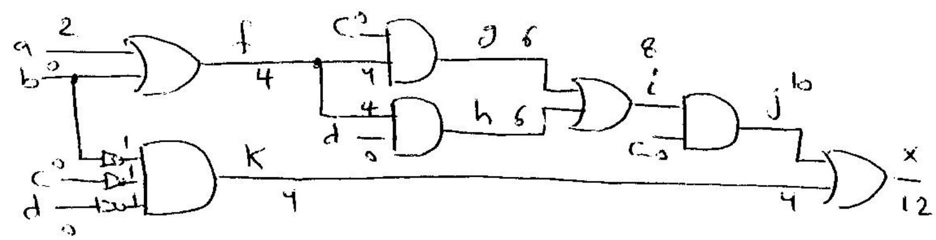
$Z = \bar{A} + Y$

Simplification using SIS:

```
sis> read_eqn hw3q3.eqn
sis> print
    {Z} = a' + y
    x = a b
    y = a b c x + a b' c' x'
sis> full_simplify
sis> print
    {Z} = a' + y
    x = b
    y = c x + c' x'
sis>
```

Q4

(i)



The data ready times are shown in the figure.

The maximum propagation delay is 12.

To compute the slack for each node, the required time for x is set to 12.

$\bar{t}_x = 12$, $\quad S_x = 12 - 12 = 0$

$\bar{t}_j = 12 - 2 = 10$, $\quad S_j = 10 - 10 = 0$

$\bar{t}_k = 12 - 2 = 10$, $\quad S_k = 10 - 4 = 6$

$\bar{t}_i = 10 - 2 = 8$, $\quad S_i = 8 - 8 = 0$

$\bar{t}_e = 10 - 2 = 8$, $\quad S_e = 8 - 0 = 8$

$\bar{t}_g = 8 - 2 = 6$, $\quad S_g = 6 - 6 = 0$

$\bar{t}_h = 8 - 2 = 6$, $\quad S_h = 6 - 6 = 0$

$\bar{t}_f = \min(6-2, 6-2) = 4$, $\quad S_f = 4 - 4 = 0$

$\bar{t}_a = 4 - 2 = 2$, $\quad S_a = 2 - 2 = 0$

$\bar{t}_b = \min(4-2, 10-3-1) = 2$, $\quad S_b = 2 - 0 = 2$

$\bar{t}_c = \min(6-2, 10-3-1) = 4$, $\quad S_c = 4 - 0 = 4$

$\bar{t}_d = \min(6-2, 10-3-1) = 4$, $\quad S_d = 4 - 0 = 4$

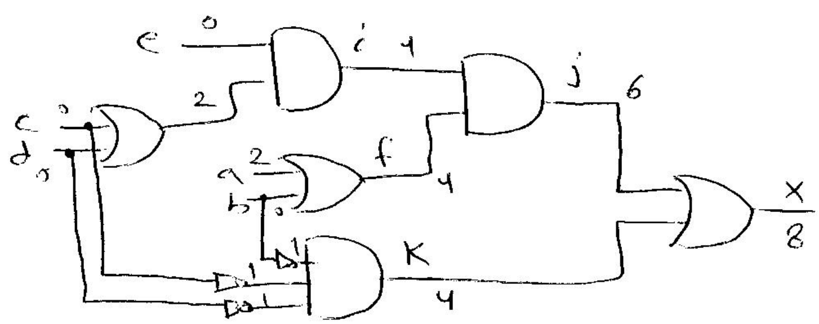
(ii) Maximum propagation delay = 12

Topological critical paths:

{a, f, g, i, j, x}, {a, f, h, i, j, x}

(iii) Since a is the one affecting the critical path, we need to bring it closer to the output.

Note that f = a + b can be factored from i. Then, f and e can be exchanged to bring a closer to the output. The resulting network is:



The delay of x is reduced from 12 to 8.

At the same time, the area of the circuit has improved