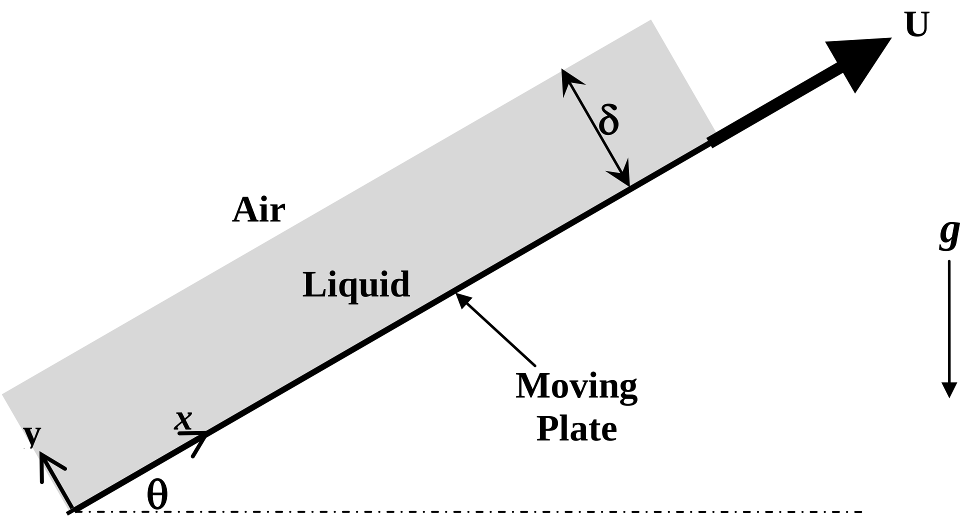
Drag-Gravity Driven Flow

A coating experiment involving a flat photographic film that is being pulled up from a processing bath by rollers with a steady velocity U at an angle θ to the horizontal. As the film leaves the bath it entrains some liquid, and in this particular experiment it has reached the stage where: The velocity of the liquid in contract with the film is $u_x = U$ at $y = 0$, the thickness of the liquid is constant at a value δ and there is no net flow of liquid (as much is being pulled up by the film as is falling back by gravity). (Clearly, if the film were to retain a permanent coating, a net upwards flow of liquid would be needed). Perform the following tasks:

1. Derive an expression for the pressure p as a function of $y, p, \delta, g,$ and θ, and hence demonstrate that $\partial p/\partial x = 0$. Assume that the pressure in the surrounding air is zero (gauge) everywhere.
2. Derive an expression for the liquid velocity u_x as a function of U, y, δ, and α, where $\alpha = \text{pg} \sin \theta/\mu$.
3. Derive an expression for the net liquid flow rate Q (per unit width) in terms of U, δ, and α. Noting that $Q = 0$, obtain an expression for the film thickness δ in terms of U and α.
4. Sketch the velocity profile $u_{x,}$ labeling all important features.

Assumptions: The following assumptions are reasonable:

1. The flow is steady and Newtonian, with constant density p and viscosity μ.
2. The z direction, normal to the plane of the diagram, may be disregarded entirely. Thus, not only is u_z zero, but all derivatives with respect to z, such as $\partial u_x/\partial z$, are also zero.
3. There is only one nonzero velocity component, namely, that in the direction of the motion of the photographic film, u_x. Thus, $u_y = u_z = 0$.
4. Gravity acts vertically downwards.



Solution:

For incompressible fluid, the density is constant:

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0$$

Since the flow is in the x-direction only, then we have only one component of the velocity $v_x \neq 0$ and $v_y = v_z = 0$, continuity equation simplifies to:

$$\frac{\partial v_x}{\partial x}=0$$

Conclusion the simplified continuity equation implies that v_x is not a function of x

$$v_x \neq f(x)$$

Also, for wide plates in z-direction, $v_x \neq f(z)$

Second we use the Navier-Stokes equations in Cartesian coordinates:

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\rho g_x,$$

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)+\rho g_y,$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z.$$

To simplify Navier-Stokes equations we can utilize the following results:

1. Flow is not pressure driven in x-direction $\frac{\partial p}{\partial x}=0$.
2. Steady state: $\frac{\partial(\text{any thing})}{\partial t}=0$
3. Plates are wide in *z*-direction: $\frac{\partial(\text{any thing})}{\partial z}=0$
4. We have one component of the velocity $v_x \neq 0$ and $v_y = v_z = 0$
5. $v_x \neq f(x, z)$ it is only a function of y: $v_x = f(y)$.
6. $g_x = g_z = 0$. Only we have gravity in y direction $g_y = -g$.

Therefore the N-S equations simplify to:

$$0 = \mu \frac{d^2 v_x}{dy^2} + \rho g_x \qquad (1)$$

$$0 = -\frac{dp}{dy} + \rho g_y \qquad (2)$$

$$0 = 0 \qquad (3)$$

Pressure Profile:

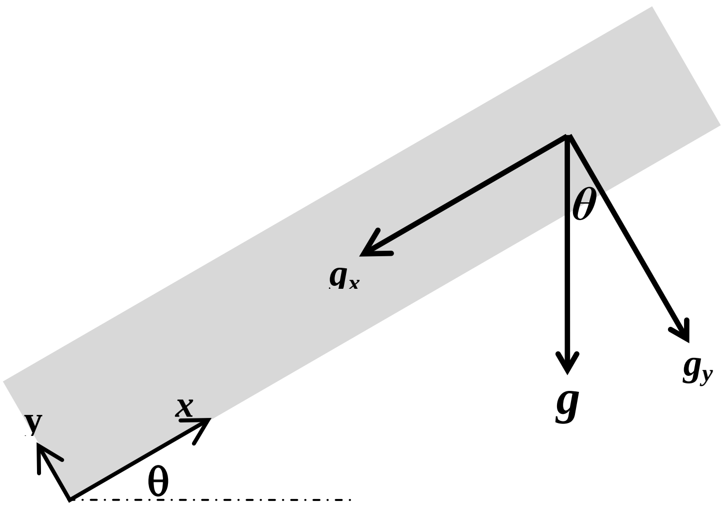
Integrate equation (2):

$$\frac{dp}{dy} = \rho\, g_y = -\rho\, g\, \mathrm{Cos}(\theta) \qquad \Rightarrow \qquad p = -\rho\, g\, \mathrm{Cos}(\theta) y + C$$

Pressure boundary condition: $y = \delta$ $\qquad p = p_{atm} = 0$ (gauge)

→ $C = \rho g \cos(\theta)\delta$

→ $p = -\rho\, g\, \mathrm{Cos}(\theta)\, (y - \delta)$



Velocity Profile:

Recall equation (1): $$0 = \mu \frac{d^2 v_x}{dy^2} - \rho g \, \mathrm{Sin}(\theta)$$

$$\frac{d^2 v_x}{dy^2} = \frac{\rho}{\mu} g \, \mathrm{Sin}(\theta) = \alpha$$

Integrate twice:

$$v_x = \frac{\alpha}{2} y^2 + C_1 y + C_2$$

To find the constants of integration apply the following Boundary Conditions:

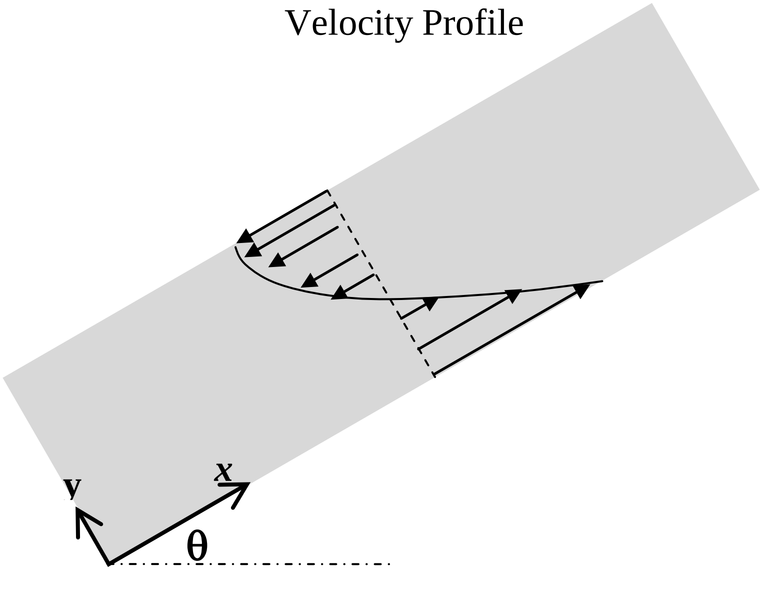
$y = 0 \quad v_x = U \quad$ (No - Slip Boundary Condition)

$y = \delta \quad \mu \frac{dv_x}{dy} = 0 \quad$ (Shear Stress between liquid and air is negligible)

$$\Rightarrow \begin{cases} C_1 = -\alpha \, \delta \\ C_2 = U \end{cases}$$

$$v_x = \frac{\alpha}{2} y^2 - \alpha \, \delta \, y + U$$
$$= U - \alpha \, y \left(\delta - \frac{y}{2} \right)$$

Velocity Profile



Volumetric Flow Rate:

$$Q = \int_A v_x \, dA$$

In this problem:

$$dA = dy\,dz$$

$$Q = \int_0^W \int_0^\delta v_x dy\,dz$$

If we consider the volumetric flow rate per unit width $W = 1$:

$$\begin{aligned} Q &= \int_0^\delta v_x dy \\ &= \int_0^\delta \left(\frac{\alpha}{2} y^2 - \alpha\,\delta\,y + U \right) dy \\ &= \frac{\alpha}{6}\delta^3 - \alpha\delta\frac{\delta^2}{2} + U\delta = U\delta - \frac{\alpha}{3}\delta^3 \end{aligned}$$

What is the value of δ such that the net flow rate is zero? In other words the quantity of the fluid going up is equal to the fluid going down?

In this case the volumetric flow rate per unit width is zero $Q = 0$:

$$Q = U\delta - \frac{\alpha}{3}\delta^3 = \delta\left(U - \frac{\alpha}{3}\delta^2 \right) = 0$$

$$\Rightarrow \quad U - \frac{\alpha}{3}\delta^2 = 0$$

$$\Rightarrow \delta = \sqrt{\frac{3U}{\alpha}}$$

Shear Stress:

$$\begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} = \mu \begin{bmatrix} \left(2\frac{\partial v_x}{\partial x} \right) & \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) & \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) \\ \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) & \left(2\frac{\partial v_y}{\partial y} \right) & \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) \\ \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right) & \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right) & \left(2\frac{\partial v_z}{\partial z} \right) \end{bmatrix}$$

Recall simplifications:

1. $\frac{\partial(\text{any thing})}{\partial z} = 0$
2. $v_x \neq 0$ and $v_y = v_z = 0$
3. $v_x \neq f(x, z)$ it is only a function of y: $v_x = f(y)$.

This leads to the following simplifications:

$$\begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} = \mu \begin{bmatrix} 0 & \left(\frac{\partial v_x}{\partial y}\right) & 0 \\ \left(\frac{\partial v_x}{\partial y}\right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, the only nonzero stresses are: $\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v_x}{\partial y}\right)$.

Recall, $v_x = \frac{\alpha}{2} y^2 - \alpha\,\delta\, y + U \qquad \Rightarrow \tau_{yx} = \mu\alpha(y - \delta)$

Shear Stress Profile