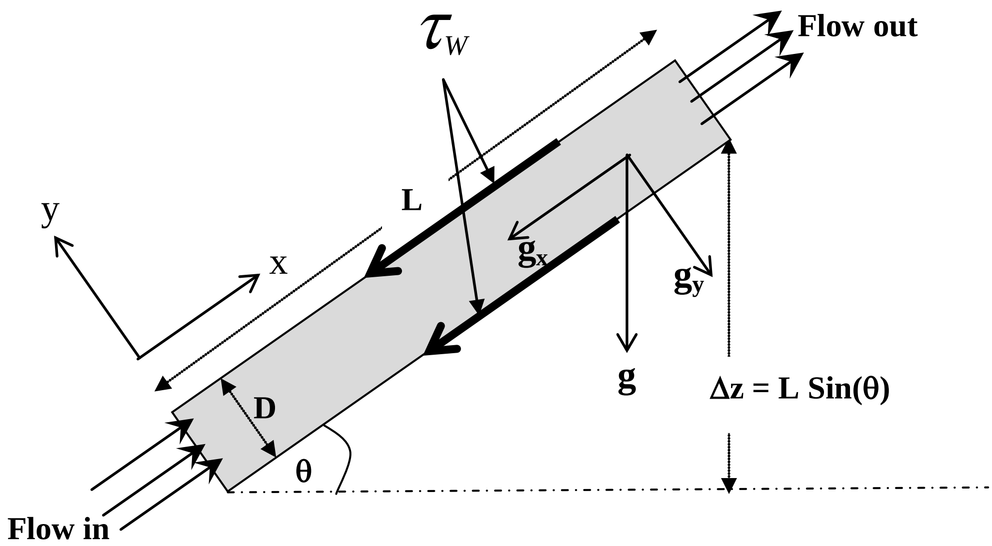
Governing Equations: Fluid Friction in Pipes



Continuity Equation (CE):

$$\rho\, u_1 A_1 = \rho\, u_2 A_2$$

$$A_1 = A_2 \qquad \Rightarrow \quad u_1 = u_2$$

Mechanical Energy Balance (MEB):

$$\Delta\left(\frac{u^2}{2}\right) + g\Delta z + \frac{\Delta P}{\rho} + w_s + \Im = 0$$

$$u_1 = u_2,\ w_s = 0 \qquad\qquad \Rightarrow g\Delta z + \frac{\Delta P}{\rho} + \Im = 0$$

$$\boxed{\Rightarrow \Im = -g\Delta z - \frac{\Delta P}{\rho}} \qquad (1)$$

Momentum Balance in *x*-Direction:

$$\left(\rho u_1^2 A_1 - \rho u_2^2 A_2 + \sum F\right)_x = 0$$

$$A_1 = A_2, u_1 = u_2 \qquad\qquad \Rightarrow \left(\sum F\right)_x = 0$$

Forces acting on the system are:

- Pressure forces.
- Friction force at the wall.
- Gravity forces.

$$\left(\sum F\right)_x = \left(P_1 A_1 - P_2 A_2\right) + \left(-\tau_W A_W\right) + \left(M_f g_x\right) = 0$$

$$A_1 = A_2 = \frac{\pi}{4} D^2$$
$$A_W = \pi D L$$
$$M_f = \frac{\pi}{4} D^2 L \rho$$
$$g_x = -g\ Sin(\theta)$$

$$\left(P_1 - P_2\right)\frac{\pi}{4} D^2 + \left(-\tau_W \pi D L\right) + \left(-\frac{\pi}{4} D^2 L\ \rho\ g\ Sin(\theta)\right) = 0$$

Simplify:
$$\frac{\left(P_1 - P_2\right)}{\rho} + \left(-\frac{4L}{D}\frac{\tau_W}{\rho}\right) + \left(-\overbrace{L\ Sin(\theta)}^{\Delta z}\ g\right) = 0$$

$$\boxed{\Rightarrow \frac{\Delta P}{\rho} = -\frac{4L}{D}\frac{\tau_W}{\rho} - g\Delta z} \qquad (2)$$

Substitute equation (2) into equation (1):

$$\boxed{\Rightarrow \Im = \frac{4L}{D}\frac{\tau_W}{\rho}} \qquad (3)$$

The wall shear stress in a pipe depends on the fluid velocity, pipe diameter, wall roughness, fluid density and viscosity:

$$\tau_W = f\left(u, D, \varepsilon, \rho, \mu\right)$$

By dimensional analysis it can be shown that the following dimensionless groups can be derived:

$$\pi_1 = \mathrm{f}_F = \frac{\tau_W}{\frac{1}{2}\rho\, u^2} \qquad \left(\text{Fanning Friction Factor}\right)$$

$$\pi_2 = \frac{\varepsilon}{D} \qquad \left(\text{Roughness Ratio}\right)$$

$$\pi_3 = \frac{\rho\, u\, D}{\mu} \qquad \left(\text{Rynold's Number}\right)$$

$$\mathrm{f}_F = f\left(\frac{\varepsilon}{D}, \mathrm{Re}\right)$$

From Experimental results the following chart and equations are available:

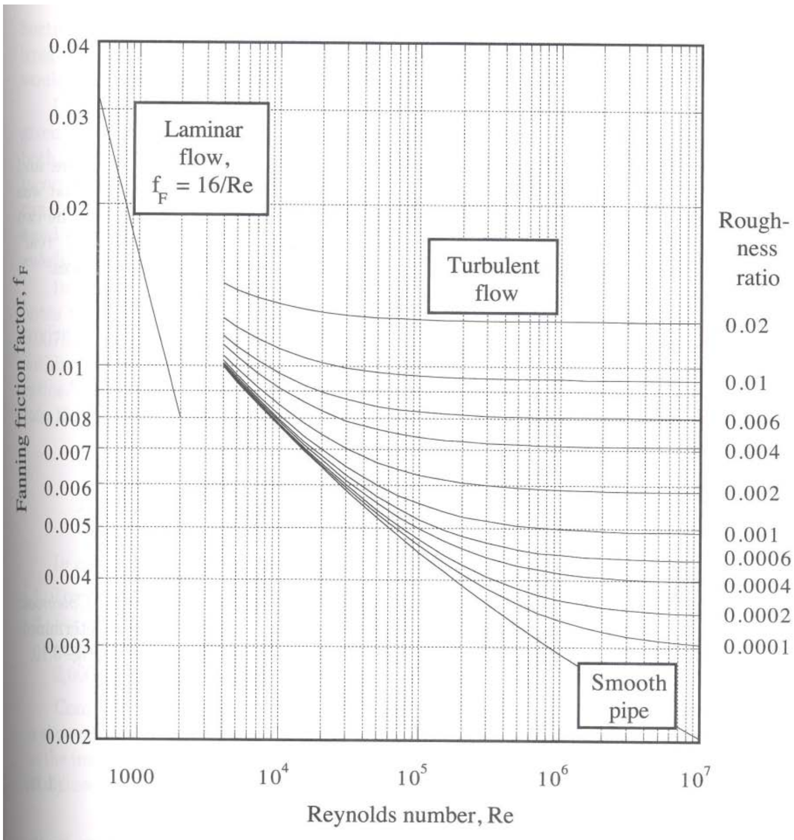


Fig. 3.10 Fanning friction factor for flow in pipes. The turbulent region is based on the Colebrook and White equation.

For laminar flow $(\mathrm{Re} < 2000)$ $\quad f_F = \dfrac{16}{\mathrm{Re}}$

For Tubulent flow $(\mathrm{Re} > 4000)$ $\quad \dfrac{1}{\sqrt{f_F}} = -1.737 \ln\left(0.269 \dfrac{\varepsilon}{D} + \dfrac{1.257}{\mathrm{Re}\sqrt{f_F}}\right)$

For turbulent flow the formula by Colebrook and White is quite complicated due to implicit nature. An alternative formula by Olujic' is available:

For Tubulent flow $(\mathrm{Re} > 4000)$ $\quad f_F = \dfrac{1}{\left\{-1.737 \ln\left(0.269 \dfrac{\varepsilon}{D} - \dfrac{2.185}{\mathrm{Re}} \ln\left(0.269 \dfrac{\varepsilon}{D} + \dfrac{14.5}{\mathrm{Re}}\right)\right)\right\}^2}$

Recall equation (3):

$$\Im = \frac{4L}{D}\frac{\tau_W}{\rho} = \frac{4L}{D}\frac{\tau_W}{\frac{1}{2}\rho u^2}\left(\tfrac{1}{2}u^2\right) = 2u^2 \frac{L}{D} f_F$$

Substitute the results in equation (1):

$$\Rightarrow -\frac{\Delta P}{\rho} = g\Delta z + \Im, \qquad \Im = 2u^2 \frac{L}{D} f_F$$

Final Result:

$$\underbrace{-\Delta P}_{\text{Pressure Drop In Pipe}} = \underbrace{\rho\, g\, \Delta z}_{\text{Pressure Drop due to Gravity}} + \overbrace{2\,\rho\, u^2 \frac{L}{D} \mathrm{f}_F}^{\text{Pressure Drop due to Friction}}$$

Summary:

- For laminar flow $(\mathrm{Re} < 2000)$

$$-\Delta P = \rho\, g\, \Delta z + 2\,\rho\, u^2 \frac{L}{D} \frac{16}{\mathrm{Re}}$$

- For tubulent flow $(\mathrm{Re} > 4000)$

$$-\Delta P = \rho\, g\, \Delta z + 2\,\rho\, u^2 \frac{L}{D} \frac{1}{\left\{-1.737 \ln\left(0.269 \frac{\varepsilon}{D} - \frac{2.185}{\mathrm{Re}} \ln\left(0.269 \frac{\varepsilon}{D} + \frac{14.5}{\mathrm{Re}}\right)\right)\right\}^2}$$