INTRODUCING NON-NEWTONIAN FLUID MECHANICS COMPUTATIONS
With Mathematica in the Undergraduate Curriculum

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A non-Newtonian fluid has a viscosity that changes with the applied shear force. These fluids are characterized by measuring or computing several rheological properties, such as the viscosity and the first and second normal stresses. Rheometers are used, under oscillatory shear flow or extensional flow, to obtain experimental values of these rheological properties while kinetic theory calculations using dumbbells allow the prediction of these rheological properties. For a Newtonian fluid (such as water), the viscosity is independent of how fast you are stirring it. For a non-Newtonian fluid the viscosity is dependent. It gets easier or harder to stir faster for different types of non-Newtonian fluids. By adding corn starch to water, one obtains a non-Newtonian fluid. Applying agitation with a spoon makes the fluid behave like a solid. Thus, the shear-thickening property of this non-Newtonian fluid becomes apparent. When agitation is stopped and the fluid is allowed to rest for a certain period of time, it recovers its liquid-like behavior.

Non-Newtonian fluids display many peculiar phenomena that can serve as the basis for multiple “fun” experiments students can perform in the laboratory. These include dye swelling, rod climbing, and suspensions of particles behavior while moving in non-Newtonian vs. Newtonian fluids. Students can determine the terminal fall velocity and rotation direction of a single settling particle as well as wall effects and interaction between particles. Problems involving non-Newtonian fluid flow are ubiquitous in modern industry, such as in polymer processing plants. The study of body fluids such as blood, which is non-Newtonian, has important applications in biomedical engineering. In the present paper, we show how one can use the mathematical software Mathematica to...
solve some simple non-Newtonian fluid problems. The most relevant Mathematica commands are inserted in the text and can be found in any introductory book such as Mathematica: A System for Doing Mathematics by Computer by Stephen Wolfram. We start by reminding the reader of the few simple constitutive equations for the power-law, Carreau, and Bingham fluids. Then, we give the velocity profile for the horizontal flow of power-law and Carreau fluids in a pipe and an annulus. The velocity profile for the fall of a Bingham liquid film is obtained in the next section. We also derive volumetric flow rate expressions for pipe flow of Bingham and power-law fluids. In the last part of the paper, we make a model determination using previously found volumetric flow rate expressions and representative data.

CONSTITUTIVE EQUATIONS FOR NON-NEWTONIAN FLUIDS

For Newtonian fluids, the shear stress, \( \tau \), is proportional to the strain rate, \( \dot{\gamma} \):

\[
\tau = \eta \dot{\gamma}
\]  

(1)

where the viscosity, \( \eta \), the proportionality factor, is constant. The situation is different for non-Newtonian fluids, and the viscosity is a function of the strain rate:

\[
\tau = \eta(\dot{\gamma}) \dot{\gamma}
\]  

(2)

Different constitutive equations, giving rise to various models of non-Newtonian fluids, have been proposed in order to express the viscosity as a function of the strain rate. In power-law fluids, the following relation is satisfied:

\[
\eta = \eta_0 \dot{\gamma}^{\alpha-1}
\]  

(3)

Dilatant fluids correspond to the case where the exponent in Eq. (3) is positive (\( n > 1 \)) while pseudo-plastic fluids are obtained when \( n < 1 \). We see that viscosity decreases with strain rate for \( n < 1 \), which is the case for pseudo-plastic fluids, also called shear-thickening fluids. On the other hand, dilatant fluids are shear-thickening. If \( n = 1 \), one recovers the Newtonian fluid behavior.

The Carreau model describes fluids for which the viscosity presents a plateau at low and high shear rates separated by a shear-thinning region:

\[
\frac{\eta - \eta_\infty}{\eta_0 - \eta_\infty} = \frac{1}{1 + (\lambda \dot{\gamma})^{2(1-n)}}
\]  

(4)

where \( \eta_0 \) is the zero-shear viscosity and \( \eta_\infty \) is the infinite-shear viscosity.

Finally, the Bingham model is defined as follows:

At low shear rates:

\[
\frac{1}{2} (\tau : \tau) \leq \tau_0, \quad \dot{\gamma} = 0
\]  

(5)

At high shear rates:

\[
\frac{1}{2} (\tau : \tau) > \tau_0^2, \quad \tau = \frac{\tau_0}{\dot{\gamma}^2}
\]  

(6)

HORIZONTAL FLOW OF CARREAU AND POWER-LAW FLUIDS IN A PIPE

**Problem Statement**

Find the velocity profiles for the laminar flow of power-law and Carreau fluids in a pipe, shown in Figure 1. Use the following values for the pressure difference \( \Delta P \), the exponent \( n \), the Newtonian fluid viscosity \( \eta_n \), the consistency index \( \kappa \), the infinite-shear viscosity \( \eta_\infty \), the zero-shear viscosity \( \eta_0 \), the relaxation parameter \( \lambda \), the pipe length \( L \), and radius \( R \), whose units appear under “Nomenclature” at the end of this article:

\( \Delta P = 100; \quad L = 50; \quad R = 0.02 \)

Newtonian fluid: \( \eta_n = 8.9 \times 10^4 \)

Dilatant fluid: \( n = 3.39 \) and \( \kappa = 10^6 \)

Pseudo-plastic fluid: \( n = 0.4 \) and \( \kappa = 5 \times 10^3 \)

Carreau fluid: \( n = 0.5, \lambda = 0.2, \eta_\infty = 1.72 \times 10^3 \), and \( \eta_0 = 0 \)

**Solution**

This problem is treated using Polymath, a numerical computational package, in Problem Solving in Chemical Engineering with Numerical Methods by Cutlip and Shacham. The governing equation is the z-component of the equation of motion in cylindrical coordinates:

\[
\frac{1}{r} \frac{d}{dr} \left[ r \left( -\frac{dv_z}{dr} \right) \right] = \frac{\Delta P}{L}
\]  

(7)

Eq. (7) is subject to the following split boundary conditions:

At \( r = 0 \): \( \tau_n = 0 \)

(8)

At \( r = R \): \( v_z = 0 \)

(9)

These kinds of mathematical problems often require the use of a particular numerical approach called the shooting technique. This method consists of guessing different values of \( v_z \) at \( r = 0 \), solving the differential equation, and checking that the no-slip boundary condition at \( r = R \) is satisfied. An analytical solution is possible for power-law fluids and details about its derivation can be found in Fluid Mechanics for Chemical Engineers by Wilkes:

\[
v_z(r) = \left( \frac{\Delta P}{L} \frac{1}{2 \kappa} \left[ \frac{1}{n+1} \right] \right) \frac{1}{n+1} \left( \left[ \frac{1}{n+1} \right] - \left[ \frac{1}{n+1} \right] \right)
\]  

(10)

---

**Figure 1. Flow of Carreau and power-law fluids in a pipe.**

Chemical Engineering Education
For the Carreau fluid, one must use a numerical approach since no analytical solution is available.

For the power-law fluids, the following Mathematica commands are used to find the velocity:

\[
\text{system}[\Omega_] := \{ \frac{\partial \tau_\nu}{\partial r}, \{r, 1\} \} == \frac{\Delta P}{L r}, \\
D[v_\nu[r], \{r, 1\}] == \frac{1}{\rho} \frac{\partial v_\nu[r]}{\partial r}\text{ or } \frac{\partial v_\nu[r]}{\partial r^2} = 0, -\frac{\tau_\nu[r]}{\rho} + (1/n) (\tau_\nu[r]/\kappa) \text{ or } \tau_\nu[r] = 0, v_\nu[r] = 0; \}
\]

\[
\text{myODEsoln}[\Omega_] := \text{NDSolve}[\text{system}[\Omega], \{v_\nu[r], \tau_\nu[r]\}, \{r, 10^{-5}, 5\}];
\]

\[
\text{yend}[\Omega_?\text{NumericQ}] := \text{Flatten}[\{v_\nu[r], \text{myODEsoln}[\Omega]\}; \\
bc = \text{FindRoot}[\text{yend}[\Omega] == 0, \{\Omega, 0, 0.5\}][[1, 2]]
\]

The graphical capability of Mathematica allows the student to plot the velocity profile without having to use different software. Figure 2 shows the velocity profile for the Newtonian, dilatant, Carreau, and pseudo-plastic cases using the commands:

![Graph showing velocity profiles of dilatant, pseudo-plastic, Carreau, and Newtonian fluids in a pipe.](attachment:image)

**Figure 2. Velocity profiles of dilatant, pseudo-plastic, Carreau, and Newtonian fluids in a pipe.**

These profiles are obtained under equal volumetric flow conditions. The velocity near the wall is higher for Carreau and pseudo-plastic fluids than for Newtonian and dilatant fluids. This results in higher heat transfer rates due to a higher convection. The approach to solve split boundary problems using Mathematica is more systematic than the one proposed by Cutlip and Shacham[4] using Polymath, despite a steeper initial learning curve for students. In fact, it automatically finds the velocity at the center of the pipe by verifying the no-slip boundary condition and using the Mathematica command FindRoot.

**HORIZONTAL FLOW OF A CARREAU AND A POWER-LAW FLUID IN AN ANNULUS**

**Problem Statement**

Find the velocity profiles for the laminar flow of power-law and Carreau fluids in an annulus, shown in Figure 3. Use the following values, where \( R_1 \) and \( R_2 \) are the inner and outer radii, and all other symbols have already been defined:

\[
\Delta P = 100; \quad L = 50; \quad R_1 = 0.02 \text{ and } R_2 = 0.05
\]

Newtonian fluid: \( \eta = 8.9 \times 10^4 \)

Dilatant fluid: \( n = 1.2 \text{ and } \kappa = 4.7 \times 10^4 \)

Pseudo-plastic fluid: \( n = 0.5 \text{ and } \kappa = 4.5 \times 10^3 \)

Carreau fluid: \( n = 0.5, \lambda = 0.2, \eta_\nu = 2.04 \times 10^3 \text{ and } \eta_\lambda = 0 \).

**Solution**

Cutlip and Shacham[4] have solved this example using Polymath. The governing equation is again the z-component of the equation of motion in cylindrical coordinates:

\[
\frac{1}{r} \frac{d}{dr} \left( \frac{dv_z}{dr} \right)^n = -\frac{\Delta P}{L} \quad (11)
\]

Eq. (11) is subject to the following split boundary conditions:

\[
\text{At } r = R_1: \quad v_z = 0 \quad (12)
\]

\[
\text{At } r = R_2: \quad v_z = 0 \quad (13)
\]

To solve this problem, we make use of the shooting technique in a similar fashion as the previous example. This method works by guessing different values of \( \tau_\nu \) at \( r = R_1 \), solving the differential equation, and checking that the no-slip boundary condition at \( r = R_2 \) is satisfied. An analytical solution[6] is available for the Newtonian fluid case:

\[
v_z(r) = \left( \frac{\Delta P}{4 \eta L} \right) \left( R_2^2 - r^2 + \frac{R_2^2 - R_1^2}{\ln(R_2 / R_1)} \ln(r / R_2) \right) \quad (14)
\]

There is no analytical solution for dilatant, pseudo-plastic, and Carreau fluids, so one must resort to a numerical method.
For the power-law fluids, the following Mathematica command is used to find the velocity as a function of \( r \):

\[
\text{system}[\Omega_] := \{ D[r, \tau_x][r][r,1] == \Delta P/L \cdot r, \\
D[v][r][r,1] == \text{If}[\tau_x[0] <= 0, -\tau_x[0][r/r^{\gamma}]\sqrt{1/r^n}, \\
(\tau_x[0][r/r^{\gamma}]\sqrt{1/r^n})], \tau_x[0][1] == \Omega, v[1][1] == 0 \} ; \\
\text{myODEsoln}[\Omega_1] := \text{NDSolve}[[\text{system}][\Omega_1], \{v_x, \tau_x\}, \{r, R_1, R_2\}] \\
\text{yend}[\Omega_1?\text{NumericQ}] := \text{Flatten}[v[x][r] /. \text{myODEsoln}[\Omega_1]] / \text{r} = R_2] \\
\text{bc} = \text{FindRoot}[\text{yend}[\Omega_1] == 0, \{\Omega, -2, 2\}][[1,2]]; \\
\text{sol1} = \text{myODEsoln}[\text{bc}] \\
\text{plt1}=\text{Plot}[v[x][r] /. \text{sol1}, \{r, 0.00001, R_1\}, \text{PlotStyle} \rightarrow \text{RGBColor}[0, 0, 1]]
\]

These profiles are obtained under equal volumetric flow conditions. The velocity profiles found for all four fluids are not symmetric. In fact, they reach a maximum value close to the radial position, given by \( r = 0.033 \), slightly less than halfway from \( R_1 \) and \( R_2 \).

**VERTICAL LAMINAR FLOW OF A BINGHAM LIQUID FILM**

**Problem Statement**

Find the velocity profile for the vertical laminar flow of a Bingham fluid down the wall depicted in Figure 5. Values of the gravitational acceleration, \( g \), the density, \( \rho \), the yield stress, \( \tau_0 \), the zero-shear viscosity, \( \eta_0 \), the film thickness, \( \delta \), are given by:

\[ g = 9.81; \quad \rho = 950; \quad \tau_0 = 5; \quad \eta_0 = 0.15 \quad \text{and} \quad \delta = 0.005 \]

**Solution**

Cutlip and Shacham\(^4\) have presented a solution of this example using Polynome. The governing equation is the z-component of the equation of motion in rectangular coordinates:

\[
\frac{d\tau_{xz}}{dx} = \rho g
\]

Eq. (15) is subject to the following split boundary conditions:

\[ \begin{align*}
\text{At} & \quad x = 0: \quad \tau_{xz} = 0 \\
\text{At} & \quad x = \delta: \quad v_z = 0
\end{align*} \]

We make the same treatment as the first two problems by applying the shooting technique:

\[
\text{system}[\Omega_1] := \{ D[r, \tau_x][r][x,1] == \text{If}[\tau_x[0] <= 0, -\tau_x[0][x/r^{\gamma}]\sqrt{1/r^n}, \\
(\tau_x[0][x/r^{\gamma}]\sqrt{1/r^n})], \tau_x[0][1] == \Omega, v[0][1] == 0 \} ; \\
\text{myODEsoln}[\Omega_1] := \text{NDSolve}[[\text{system}][\Omega_1], \{v_x, \tau_x\}, \{x, 0, \delta\}] \\
\text{yend}[\Omega_1?\text{NumericQ}] := \text{Flatten}[v[x][r] /. \text{myODEsoln}[\Omega_1]] / \text{r} = R_2] \\
\text{bc} = \text{FindRoot}[\text{yend}[\Omega_1] == 0, \{\Omega, 0, 0.5\}][[1,2]]; \\
\text{sol1} = \text{myODEsoln}[\text{bc}] \\
\text{plt1}=\text{Plot}[v[x][r] /. \text{sol1}, \{x, 0, \delta\}, \text{PlotStyle} \rightarrow \text{RGBColor}[0, 0, 1]]
\]

For the Newtonian case, an analytical expression for the velocity, \( v_z \), as a function of position, \( x \), can be easily derived:

\[
v_z = \frac{\rho g}{2\eta} \left[ 1 - \left( \frac{x}{\delta} \right)^2 \right]
\]

In Figure 6, we show the velocity profile for the Newtonian and the Bingham fluids. This plot is obtained by using the Mathematica commands,
sol1 = myODEsolv[bc]
plt1 = Plot[v[x], {x, 0, δ}, PlotStyle → RGBColor[0, 0, 1]]

A comparison of the velocity profile obtained using the analytical solution for the Newtonian fluid and the velocity profile corresponding to the Bingham fluid shows that the latter is flat near the surface of the liquid film. In fact, we have a nonzero velocity gradient only when \( τ_\alpha > τ_\eta \). This behavior is typical of Bingham fluids.

**EXPRESSIONS OF VOLUMETRIC FLOW RATES**

**Problem Statement**

Derive expressions of volumetric flow rates for pipe flow of Bingham and power-law fluids using symbolic computations with Mathematica.

**Solution**

**Power-law fluid case**

First, we find the expression of the shear stress, \( τ_\alpha \), as a function of the radial position, \( r \):

\[
\text{sol3} = \text{DSolve}[D[r \tau_\alpha[r], r] == -ΔP/L \cdot r, \tau_\alpha[r], r] \quad \tau_\alpha[r] = \text{sol3}[1, 1, 2]/C[1] - 0
\]

We get the following result:

\[
τ_\alpha = -\frac{ΔPr}{2L} \tag{19}
\]

Then, we determine the velocity distribution using the symbolic command, \( \text{DSolve} \):

\[
\text{sol4} = \text{DSolve}[D[v[r], r] == (-τ_\alpha[r]/\eta) \cdot (1/n), v[r], r] \quad v[r] = \text{sol4}[1, 1, 2]/C[1] \cdot \frac{2^{-\frac{1}{n}} \cdot nR \cdot (\frac{ΔP}{L})^{\frac{1}{n}}}{1+n} \cdot \frac{1}{C[1]}
\]

Finally, the symbolic command, \( \text{Integrate} \), is used, \( Q = \text{Integrate}[2 \Pi r v[r], \{r, 0, R\}] \).

<table>
<thead>
<tr>
<th>TABLE 1</th>
<th>Volumetric Flow Rate vs. Pressure Gradient</th>
</tr>
</thead>
<tbody>
<tr>
<td>ΔP/L (Pa/m)</td>
<td>( 10^5 \times Q ) (m³/s)</td>
</tr>
<tr>
<td>10000</td>
<td>5.37</td>
</tr>
<tr>
<td>20000</td>
<td>26.4</td>
</tr>
<tr>
<td>30000</td>
<td>68.9</td>
</tr>
<tr>
<td>40000</td>
<td>129</td>
</tr>
<tr>
<td>50000</td>
<td>235</td>
</tr>
<tr>
<td>60000</td>
<td>336</td>
</tr>
<tr>
<td>70000</td>
<td>487</td>
</tr>
<tr>
<td>80000</td>
<td>713</td>
</tr>
<tr>
<td>90000</td>
<td>912</td>
</tr>
<tr>
<td>100000</td>
<td>1100</td>
</tr>
</tbody>
</table>

and we get the following expression for the volumetric flow rate,

\[
Q = \frac{2^{-\frac{1}{n}} \cdot nR \cdot (\frac{ΔP}{kL})^{\frac{1}{n}}}{1+n} \tag{20}
\]

**Bingham fluid case**

Just like the treatment above, we start by finding the expression of the shear stress, \( τ_\alpha \), as a function of the radial position, \( r \):

\[
\text{sol1} = \text{DSolve}[D[r \tau_\alpha[r], r] == -ΔP/L \cdot r, \tau_\alpha[r], r]
\]

\( τ_\alpha[r] = \text{sol1}[1, 1, 2]/C[1] - 0 \)

We get the following result:

\[
τ_\alpha = -\frac{ΔPr}{2L} \tag{19}
\]

In the first part of the derivation, we determine the velocity distribution between \( r = (2τ_\alpha/L)/ΔP \) and \( r = R \) using boundary condition \( v(R) = 0 \) and the symbolic command, \( \text{DSolve} \):

\[
\text{sol2} = \text{DSolve}[D[v[r], r] == \frac{(τ_\alpha[r]+τ_\eta) \cdot δ}{η}, v[r], r]
\]

\( v[r] = \text{sol2}[1, 1, 2]/C[1] - (R \cdot τ_\eta)/η \)

The symbolic command, \( \text{Integrate} \), is used to obtain the expression of the volumetric flow rate between \( r = (2τ_\alpha/L)/ΔP \) and \( r = R \),

\[
Q1 = \text{Integrate}[2 \Pi r v[r], \{r, 0, 2τ_\alpha/L/ΔP\}]
\]

In the second part of the derivation, we determine the constant velocity, \( v_0 \), between \( r = 0 \) and \( r = (2τ_\alpha/L)/ΔP \) using the following symbolic command:

\[
v_0 = 1/μ \cdot (ΔP/L) \cdot (r^2 - R^2)/4 + τ_\eta \cdot μ \cdot (r-R)/r = 2τ_\alpha/L/ΔP
\]

This is nothing more than expressing the continuity of the velocity at \( r = (2τ_\alpha/L)/ΔP \). In fact, we have written that \( v_0 = v_0((2τ_\alpha/L)/ΔP) \) in the above Mathematica statement.

The symbolic command, \( \text{Integrate} \), is used to obtain the expression of the volumetric flow rate between \( r = 0 \) and \( r = (2τ_\alpha/L)/ΔP \),

\[
Q2 = \text{Integrate}[2 \Pi r v_0, \{r, 0, 2τ_\alpha/L/ΔP\}]
\]

and we get the following expression for the overall volumetric flow rate,

\[
Q = \frac{πR^3 \cdot ΔP}{8ηL} + \frac{πR^3τ_\eta}{3η} + \frac{2πτ_α^3L^3}{3η \cdot ΔP^3} \tag{21}
\]

**NON-NEWTONIAN FLUID MODEL DETERMINATION**

**Problem Statement**

Wilkes[5] provides representative values of the volumetric flow rate vs. the applied pressure gradient for horizontal flow in a pipe. These values are reproduced in Table 1. The pipe radius is equal to \( R = 0.01 \text{m} \). Use these representative values,
in conjunction with the analytical expression of the volumetric flow rates determined in the previous section, to compute the parameters of the constitutive equation.

**Solution**

First, we compute the following sum:

\[ J = \sum_{i=1}^{10} (Q_{i}^{mp} - Q_{i}^{th})^2 \]  \hspace{1cm} (22)

where \( Q_{i}^{mp} \) and \( Q_{i}^{th} \) are the representative value and analytical expression of the volumetric flow rate. Then, we use the built-in command of Mathematica, *FindMinimum*, to determine the values of \( n \) and \( \kappa \) for the power-law model, and \( \tau_0 \) and \( \eta \) for the Bingham model that minimize the objective function, \( J \). The approach used here is the least squares method. For the power-law model, we find \( n = 0.437 \) and \( \kappa = 6.708 \), while for the Bingham model the result is \( \tau_0 = 77.55 \) and \( \eta = 0.0326 \). The value of the sum given by Eq. (22) is \( 9.89 \times 10^6 \) for the Bingham model and \( 2.67 \times 10^7 \) for the power-law model. Thus, we conclude that the power-law model fits the representative data better.

**CONCLUSIONS**

We presented the solution of four non-Newtonian fluid mechanics problems using Mathematica. The velocity profile is obtained for the horizontal flow of power-law and Carreau fluids in pipes and annuli, and for the vertical laminar flow of a Bingham fluid. These problems have split boundary conditions and were solved using the shooting techniques. Analytical expressions of volumetric flow rates for pipe flow of the Bingham and power-law fluids were derived using Mathematica. The parameters of the constitutive equation of non-Newtonian fluids were obtained from representative data of flow rates measured under different applied pressure gradients in a horizontal pipe. These problems are simple enough to constitute an excellent introduction to the field of non-Newtonian fluid mechanics. Students at the National Institute of Applied Sciences in Tunis performed well despite no previous knowledge of Mathematica. Mathematica notebooks are available from author upon request or at the information center.\(^{[1]}\)

**NOMENCLATURE**

- \( g \) gravitational acceleration (m/s\(^2\))
- \( Q \) volumetric flow rate (m\(^3\)/s)
- \( L \) pipe length (m)
- \( n \) power-law exponent
- \( \Delta P \) pressure difference (Pa)
- \( R \) pipe radius (m)
- \( R_1, R_2 \) annulus radii (m)
- \( r \) radial position (m)
- \( v_0 \) velocity (m/s)
- \( z \) axial position (m)
- \( \kappa \) power-law consistency index (N · s\(^n\)/m\(^n\))
- \( \delta \) film thickness (m)
- \( \lambda \) relaxation parameter (s)
- \( \eta \) viscosity (kg/m·s)
- \( \eta_0 \) zero-shear viscosity (kg/m·s\(^2\))
- \( \eta_\infty \) infinite-shear viscosity (kg/m·s\(^2\))
- \( \rho \) density (kg/m\(^3\))
- \( \tau_0 \) yield stress (kg/m·s)
- \( \tau_\infty \) shear stress (kg/m·s)

**REFERENCES**

1. <http://library.wolfram.com/informational/search/?search_person_id=1536>