13. *Torque in a Couette viscometer—M.* Fig. P6.13 shows the horizontal cross section of a concentric cylinder or "Couette" viscometer, which is an apparatus for determining the viscosity μ of the fluid that is placed between the two vertical cylinders. The inner and outer cylinders have radii of r_1 and r_2, respectively. If the inner cylinder is rotated with a steady angular velocity ω, and the outer cylinder is stationary, derive an expression for v_θ (the θ velocity component) as a function of radial location r.

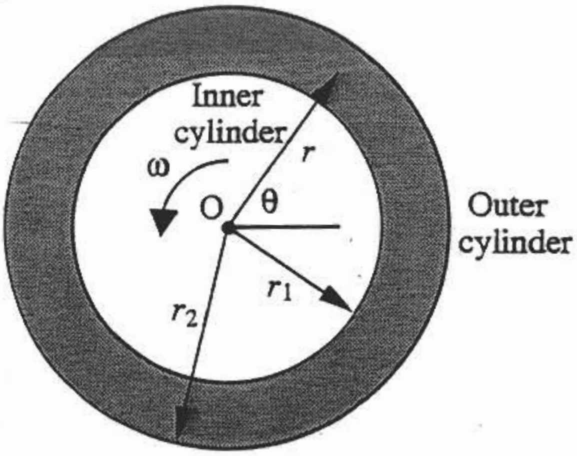

Fig. P6.13 Section of a Couette viscometer.

If, further, the torque required to rotate the *inner* cylinder is found to be T per unit length of the cylinder, derive an expression whereby the unknown viscosity μ can be determined, in terms of T, ω, r_1, and r_2. *Hint*: you will need to consider one of the shear stresses given in Table 5.8.

$$T = \text{Force} * r_1 \qquad (1)$$

$$g_z = -g \; , \; g_\theta = g_r = 0$$

To oppose the effect of the shear stress

$$\text{Force} = -\tau_{r\theta}\Big|_{r=r_1} * 2\pi r_1 L \qquad (2)$$

the surface on which the sheare stress acts

direction of the shear stress

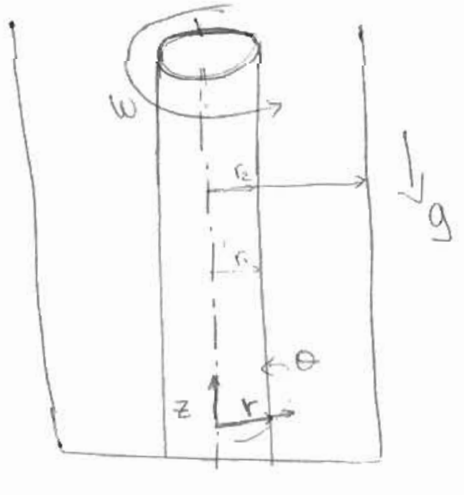

Table 5.8

$$\tau_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] \quad (3)$$

Asumptions

1) Steady State
2) Newtonian fluid (μ is constant)
3) Incompressible fluid (ρ is constant)
4) $u_z = u_r = 0$ Velocity in the θ direction
5) Axisymmetric Problem (all the variable are not $f(\theta)$)

Continuity Equation

ρ is Constant, $u_r = u_z = 0$

$$\therefore \frac{\partial u_\theta}{\partial \theta} = 0$$

$$\therefore u_\theta = f(r)$$

Momentum balance

r-direction: $$\frac{-\rho u_\theta^2}{r} = -\frac{\partial P}{\partial r} \quad (4)$$

$\frac{\partial P}{\partial \theta} = 0$

θ-direction: $$\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(u_\theta r)}{\partial r}\right)\right] = 0 \quad (5)$$

z-direction: $$0 = -\frac{\partial P}{\partial z} + \rho g_z \Rightarrow g_z = -g \quad (6)$$

$$P = f(r, z)$$

equation (6)

$$\frac{\partial P}{\partial z} = -\rho g$$

Integration gives $P = -\rho g z + f_1(r)$

Equation (4)

$$u_\theta = f(r)$$

$$P = \int \frac{u_\theta^2}{r}\, dr + f_2(z)$$

$$\therefore f_1(r) = \int \frac{u_\theta^2}{r}\, dr \quad , \quad f_2(z) = -\rho g z$$

equation (5)

$$\frac{d}{dr}\left(\frac{1}{r}\, \frac{d(r u_\theta)}{dr}\right) = 0$$

Integration $\Rightarrow \dfrac{1}{r}\, \dfrac{d(r u_\theta)}{dr} = C_1$

or $\dfrac{d(r u_\theta)}{dr} = C_1 r$

Integration $\Longrightarrow$ $r u_\theta = C_1 \dfrac{r^2}{2} + C_2$

or $u_\theta = C_1 \dfrac{r}{2} + \dfrac{C_2}{r}$

Boundary Condition

angular velocity $\frac{1}{s}$

$$\left.\begin{array}{ll} r = r_1 & u_\theta = \omega r_1 \\ r = r_2 & u_\theta = 0 \end{array}\right\} \text{No slip condition}$$

$$\therefore \omega r_1 = C_1 \frac{r_1}{2} + \frac{C_2}{r_1}$$

$$0 = C_1 \frac{r_2}{2} + \frac{C_2}{r_2}$$

Solving for C_1 & C_2 and substituting in the expression of u_θ gives

$$u_\theta = \overbrace{\frac{r_1 w}{r_1 - \frac{r_2^2}{r_1}}}^{c_1/2} r + \overbrace{\frac{r_1 w}{\left(\frac{1}{r_1} - \frac{r_1}{r_2^2}\right)}}^{c_2} \frac{1}{r}$$

Pressure

$$P = -\rho g z + \int \frac{u_\theta^2}{r}\, dr$$

$$P = -\rho g z + \frac{c_1^2}{4} r^2 + c_1 c_2 \ln r - \frac{c_2^2}{2r^2}$$

$$C_1 = \frac{-2 r_1 w}{\left(\frac{r_2^2}{r_1} - r_1\right)} \quad , \quad C_2 = \frac{r_1 w}{\left(\frac{1}{r_1} - \frac{r_1}{r_2^2}\right)}$$

equation (3)

$$\tau_{r\theta} = \mu\left[r \frac{d}{dr}\left(\frac{u_\theta}{r}\right)\right]$$

$$\frac{u_\theta}{r} = \frac{c_1}{2} + \frac{c_2}{r^2}$$

$$\frac{d}{dr}\left(\frac{u_\theta}{r}\right) = -2C_2/r^3$$

$$\therefore \tau_{r\theta} = -\frac{2\mu C_2}{r^2}$$

equation (2)

$$\text{Force} = \frac{2\mu C_2}{r_1^2} * 2\pi r_1 L = \frac{4\pi\mu C_2 L}{r_1}$$

equation (1)

$$T = 4\pi\mu L\, C_2$$

or $$T = \frac{4\pi\mu L r_1 w}{\left(\frac{1}{r_1} - \frac{r_1}{r_2^2}\right)}$$

Solving for μ gives

$$\boxed{\mu = \frac{T\left(\frac{1}{r_1} - \frac{r_1}{r_2^2}\right)}{4\pi L r_1 w}}$$