SE513: Systems Identification

Lecture 4

Random Process

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Lecture Outlines

- Examples of computing mean and variance
- Deterministic and Stochastic Signals
- Random Process
- Classification
- Stationarity
- Statistical Independence
- Time Average & Ergodicity
- Correlation functions

Example A1

- Compute mean and Variance

x : random variable

$$p(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

$$\bar{x} = E\{x\} = \int_0^{\infty} x\, p(x) dx = \int_0^1 x \left[3x^2\right] dx = \frac{3}{4}$$

$$V(x) = E\{(x-\bar{x})^2\} = E\{x^2) - \bar{x}^2 = \int_0^1 x^2 \left[3x^2\right] dx - \left(\frac{3}{4}\right)^2 = \frac{3}{5} - \frac{9}{16} = 0.0375$$

Multiple Random Variables

- X1 and X2 are two random variables

$$P(x_1, x_2) = \text{Prob}\,(\text{X}_1 \le x_1, \text{X}_2 \le x_2)$$

joint probability distribution

$$P(-\infty, -\infty) = 0$$

$$P(\infty, \infty) = 1$$

$$0 \le P(x_1, x_2) \le 1$$

Marginal Distribution

$$p_1(x_1) = p(x_1, \infty) = \int_{-\infty}^{\infty} p(x_1, x_2)\ dx_2$$

$$p_2(x_2) = p(\infty, x_2) = \int_{-\infty}^{\infty} p(x_1, x_2)\ dx_1$$

Multiple Random Variables

- X1 and X2 are two random variables

$$Cov(X_1, X_2) = E\{(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)\}$$

$$Let \;\; Y = a\,x_1 + b\,x_2$$

$$What\ is\ E\{Y\} = ?, \;\; What\ is\ \ V(Y) = ?$$

$$E\{Y\} = a\,\bar{x}_1 + b\,\bar{x}_2$$

$$V(Y) = a^2 V(X_1) + b^2 V(X_2) + ab\ Cov(X_1\, X_2)$$

Multiple Random Variables
Example

- X1 and X2 are two random variables

$$p(x_1, x_2) = \begin{cases} 3x_1 & 0 \le x_2 \le x_1 \le 1 \\ 0 & therwise \end{cases}$$

joint probability density function

$M \arg inal\ Distribution$

$$p_1(x_1) = \begin{cases} 3x_1 & 0 \le x_1 \le 1 \\ 0 & therwise \end{cases}$$

$$p_2(x_2) = \begin{cases} 1.5(1 - x_2{}^2) & 0 \le x_2 \le 1 \\ 0 & therwise \end{cases}$$

Multiple Random Variables
Example

- X1 and X2 are two random variables

$$\mu_1 = E\{x_1\} = \int_0^1 x_1(3x_1)dx_1 = 0.75$$

$$\mu_2 = E\{x_2\} = \int_0^1 x_2(1.5(1-x_2{}^2))dx_2 = 0.375$$

$$E\{X_1X_2\} = \int_0^1 \int_0^1 x_1x_2(3x_1)dx_1dx_2 = 0.3$$

$$Cov\{X_1X_2\} = E\{X_1X_2\} - \mu_1\mu_2 = 0.3 - 0.75 * 0.375 = 0.0188$$

Correlation

- X1 and X2 are two random variables

$$Cor\{X_1, X_2\} = E\{X_1 X_2\}$$

$$X_1, X_2 \ are \ uncorrelated \quad \Rightarrow \quad Cor\{X_1, X_2\} = 0$$

$$\not\Leftarrow$$

$$X_1, X_2 \ are \ independent \quad \Rightarrow \quad Cor\{X_1, X_2\} = 0$$

$$\not\Leftarrow$$

Lecture 4

Random Process

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Deterministic and Stochastic Signals

Deterministic Signals

$y(t)=10 \sin(2t)$

$x(t)=2-3t$

Random (Stochastic) Signals

$y(t)=A \sin(2t)$

A is random variable uniformly distributed [1, 2]

$x(t)=A$

A is random variable

$v(t) = 10 \sin(2t+A)$

A is normally distributed $N(0,1)$

Random Process

- A random process X(t,s) is a function of time and sample space.
- X(t,s) is often written as X(t) for simplicity
- Each member time function is called sample function , ensemble member or a realization of the process.
- When t is fixed and s varies

  $X(t,s) \rightarrow$ random variable

Example of Random Process

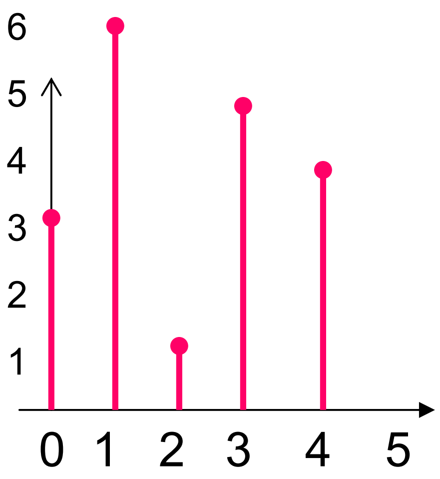
$$x(t) = 2\cos(3t + s)$$

$$s \text{ is uniformly distributed random variable on } [0, 2\pi]$$

- Random variables are functions of the sample space.
- Random processes are functions of time and the sample space

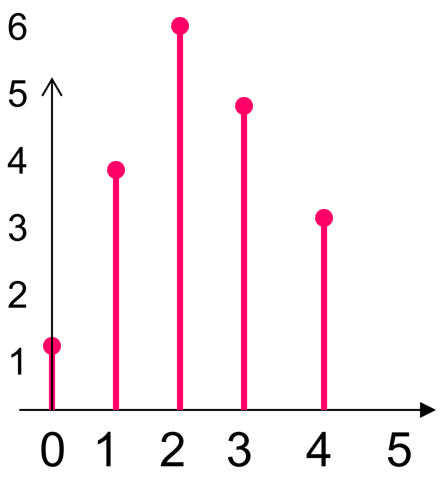
Example

Tossing a die



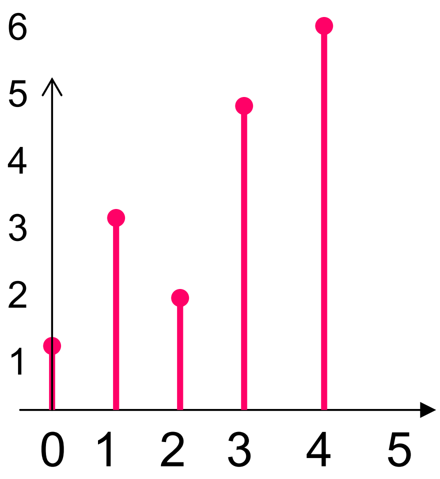
Ali

One sample signal (realization)



Ahmed

One sample signal (realization)



Samir

One sample signal (realization)

Ensemble of players. Each generates a different but statistically similar realization.

Classification of Random Processes

Continuous Random Process	Discrete Random Process
Continuous Random Sequence	Discrete Random Sequence

Classification of Processes

1. Continuous Random Process

X(t,s): X is continuous and t can have a continuum of values

Example: thermal noise

2. Continuous Random Sequence

X is continuous but the time t is discrete

Classification of Processes

3. Discrete Random Process

   X has discrete values while t is continuous

4. Discrete Random Sequence

   Both time and the random variable are discrete

Stationary

- A random Process is stationary if all its statistical properties do not change with time.
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- First order stationary process
- Second order stationary process
- Wide Sense stationary process
- Strict Sense Stationary process

First order stationary process

- The probability density function of a first order stationary process satisfies

$$p_x(x_1, t_1) = p_x(x_1, t_1 + \Delta) \quad for \; all \; t_1, \Delta \in \Re$$

- If x(t) is a first order stationary process then

$$E\{x(t)\} = E\{x(t+\Delta)\} = \bar{x} = \text{constant}$$

Second order stationary process

- The Second order density function of a second order stationary process satisfies

$$p_x(x_1, x_2, t_1, t_2) = p_x(x_1, x_2, t_1 + \Delta, t_2 + \Delta)$$

$$for\ \ all\, t_1, t_2, \Delta \in \Re$$

- If x(t) is a second order stationary process then

$$R_{xx}(t_1, t_1 + \tau\} = E\{x(t_1)\ \ x(t_1 + \tau)\} = R_{xx}(\tau)$$

R

- The second order stationary may be more restrictive than needed in many applications. A more useful form is Wide Sense Stationary process.
- A process is Wide Sense Stationary process if

$$E\{x(t)\} = \bar{x} = \text{constant}$$

$$E\{x(t_1)\ x(t_1+\tau)\} = R_{xx}(\tau)$$

- Second order stationary $\Rightarrow$ Wide Sense Stationary process.

$\not\Leftarrow$

Example

$x(t) = A\cos(\omega_0 t + \phi)$

A and ω_0 are constants,

ϕ is uniformly distributed random variable on $[0,2\pi]$

$$\mathrm{E}\{\mathrm{x(t)}\} = \int_0^{2\pi} A\cos(\omega_0 t + \phi)\ \frac{1}{2\pi}\ \mathrm{d}\phi = 0$$

$$E\{x(t)x(t+\tau)\} = \int_0^{2\pi} A\cos(\omega_0 t + \phi)\, A\cos(\omega_0 (t+\tau) + \phi)\ \frac{1}{2\pi}\ \mathrm{d}\phi = 0$$

$$= \frac{A^2}{2}\cos(\omega_0 \tau) + \frac{A^2}{2} E\{\cos(2\omega_0 t + \omega_0 \tau + 2\phi)\} = \frac{A^2}{2}\cos(\omega_0 \tau)$$

$$\left.\begin{array}{l} E\{x(t)\} = \text{constant} \\ E\{x(t)x(t+\tau)\} = R_{xx}(\tau) \end{array}\right\} \Rightarrow x(t) \text{ is wide sense stationary}$$

Nth order and strict sense Stationary

- A random process is stationary of order N if its Nth order density function is invariant to shift in the origin time.
- Stationarity of order N means stationarity of orders k for all $k \leq N$
- A random process is strict sense stationary if it is stationary of all orders N=1,2,3,….

Jointly Wide Sense stationary

- $x(t)$ and $y(t)$ are jointly wide sense stationary if each is wide sense stationary and $E\{x(t)\ y(t+\tau)\} = R_{xy}(\tau)$

Time Average

- The time average of x(t)

$$T.A. of\ x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt$$

The time autocorrelation function is

$$R_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,x(t+\tau)\,dt$$

Ergodic Process

- A process is ergodic If the time average is equal to the ensemble average.
- If a process is ergodic then a single sample time signal contains all statistical variations of the process
- We do not need more than one time sample.

Ergodic Process

- If the time average and time autocorrelation are equal to the statistical average and statistical autocorrelation then the process is ergodic.

$$\int_{-\infty}^{\infty} x(t)\,p(x)\,dx = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)dt$$

$$\int_{-\infty}^{\infty} (x(t)x(t-\tau))\,p(x)\,dx = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)\,dt$$

- Ergodity is very restrictive. The assumption of ergodity is used to simplify the problem

Reading Material

- Chapter 6 (P.Z. Peebles. Probability, Random variables and Random Signal Principles, 2nd Ed. 1987)