SE513: Modeling & Systems Identification I

Lecture 3
Review of Probability & random variables

Dr. Samir Al-Amer

Lecture Outlines

- Probability
- Probability of multiple events
  - Independent events
  - Mutually exclusive events
  - Non- Mutually exclusive events
- Mean and variance
- Continuous variable

Probability

- Let
  - X be an event occurring as a result of some experiments or trials
  - n_x : # of trials resulting in X
  - n : total # of trials

$$\text{Probability of } X = p(X) = \lim_{n\to\infty} \frac{n_x}{n}$$

Conduct an infinite number of trials

$$\text{Probability of } X = \frac{\text{\# of tirals resulting in X}}{\text{total number of trials}}$$

Probability

Alternative definition/ more practical

- Let
  - X be an event occurring as a result of some experiments or trials
  - N : # of all possible outcomes (size of sample space)
  - N_x : # of outcomes favorable to X

$$\text{Probability of } X = p(X) = \frac{N_x}{N}$$

  - Tossing a fair coin
  - Two possible outcomes { H,T}
  - X= getting a head {H} in first trial
  - N=2, Nx=1 p(X)=0.5

Probability Example

- When a die is tossed
- Sample space ={1,2,3,4,5,6}
- Let X be the occurrence of ‘4’
- N=6
- Nx=1
- p(4)=1/6

Probability Example

- When a die is tossed **once**
- Sample space ={1,2,3,4,5,6}
- Let X be the occurrence of an even number (2 , 4 or 6)
- N=6
- Nx=3
- p(even)=3/6 =0.5

Probability of Multiple Events

- Independent events
- Mutually exclusive events
- Non- Mutually exclusive events

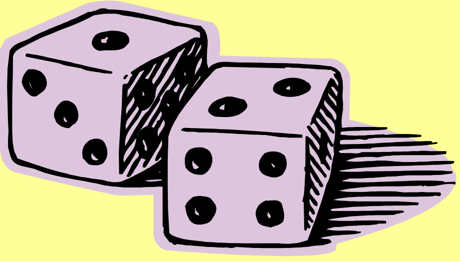
Probability of Independent Events

- Two events X and Y are **independent** if the occurrence of X does not affect the probability of Y and vice versa

- If X and Y are independent then

$p(X\ Y)=p(X)\ p(Y)$

Probability of X and Y

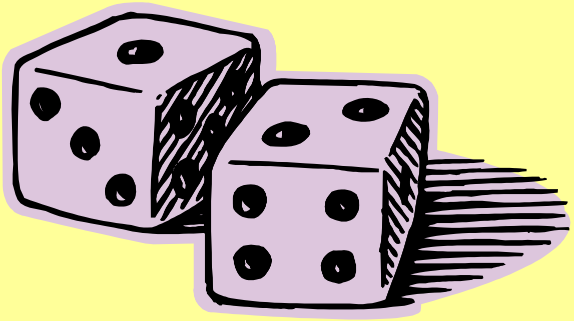


Probability of Independent Events

Two dice are tossed.

What is the probability that ‘3’ appears twice?

- X : ‘3’ appears at the top of the first die
- Y : ‘3’ appears at the top of the second die
- X and Y are independent



P(X Y)=P(X) P(Y)

P(3)=1/6

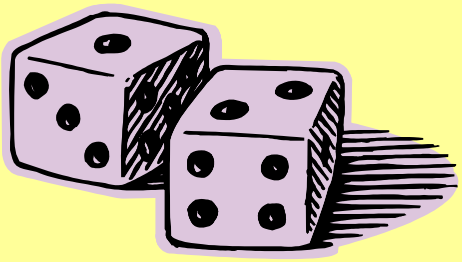
P(3 3)=P(3)P(3)=1/36

Probability of Independent Events

- If X_1 , $X_{2,\ldots,}$ X_m are **independent**

then

$$P(X_1 X_2 \ldots X_m)=P(X_1) P(X_2) \ldots P(X_m)$$



Notation

$$p(X|Y) = \text{probability of occurrence of event X knowing that Y occurred}$$

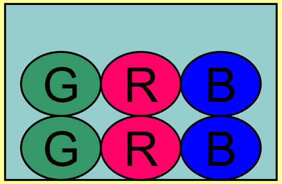
If X and Y are independent $\Rightarrow$

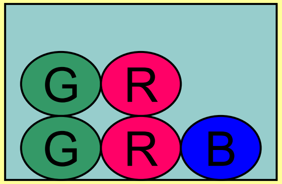
$$p(X|Y) = p(X)$$

$$p(Y|X) = p(Y)$$

Probability of Dependent Events

- $p(XY)=p(X)p(Y|X)$
- Probability of X and Y is the product of probability of X times the probability of Y given that X occurred
- X: first ball taken is blue
- Y: Second ball taken is blue
- $P(X)=2/6$
- $P(Y|X)=1/5$
- $P(XY)=(2/6)(1/5)$





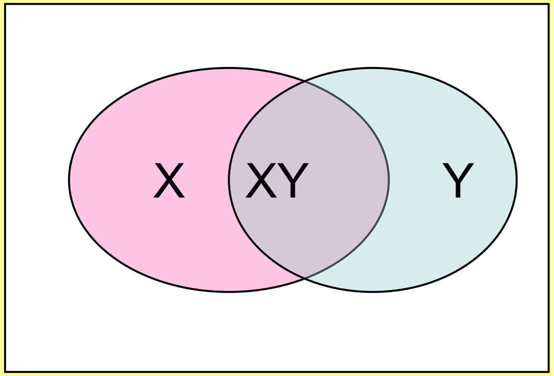
Probability of Mutually Exclusive Events

- Two events X and Y are **mutually exclusive** if the occurrence of X precludes the occurrence of Y and vice versa.
- X and Y can not occur at the same time
- p(X+Y): the probability of occurrence of X or Y
- p(X+Y)= p(X)+ p(Y)

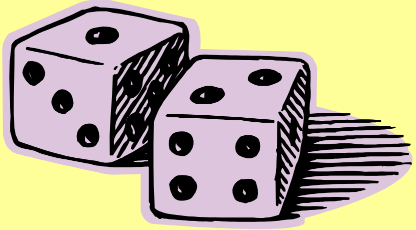
Probability of Non-Mutually Exclusive Events

If X and Y are non-mutually exclusive then

$$p(X+Y)=p(X)+p(Y)-p(XY)$$



What is the probability one '2' appears in two-dice toss?



Alternative Notation

$$p(XY) = p(X \cap Y)$$

$$P(XYZ) = P(X \cap Y \cap Z)$$

$$p(X+Y) = p(X \cup Y)$$

$$p(X+Y+Z) = p(X \cup Y \cup Z)$$

DeMorgan's Laws

$$\overline{(X \cup Y)} = \overline{X} \cap \overline{Y}$$

$$\overline{(X \cap Y)} = \overline{X} \cup \overline{Y}$$

$$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$$

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

Random Variables

Random Variables

- Random variables are real valued functions whose domain is the sample space.

Example:

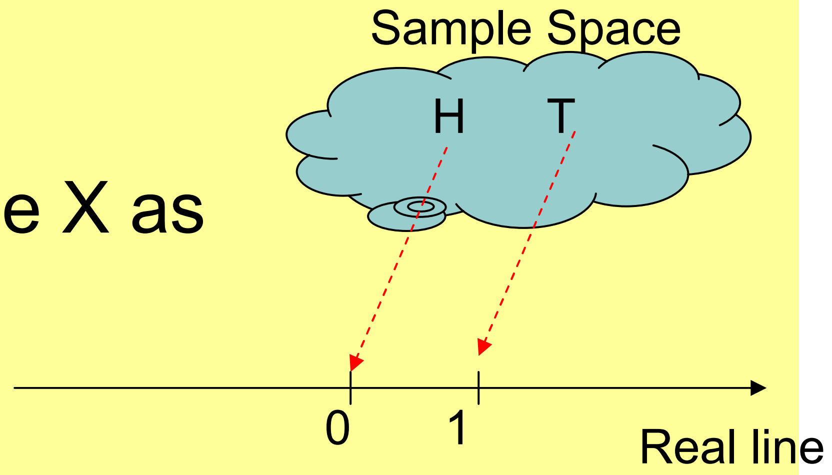
Toss a coin

Sample Space={H,T}

Define the random variable X as

X(H)=0

X(T)=1



Random Variables

Notation

- Capital letters are used for Random variables
- Lower case letters are used for the values a random variable takes.

Example: $p(X=x)$

The probability that the random variable X is equal to x.

$p(X=1)=?$ 0.5 if we have a fair coin

Random Variables

Example:

4 dice are tossed

x = # of appearance of '5' on the top of four dice toss.

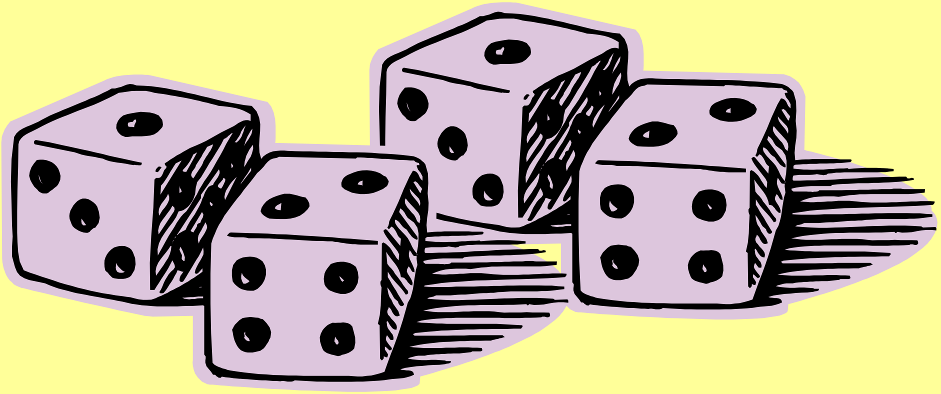
There are four possible values that x can take 0,1,2,3 and 4

X is a random variable. It is a number that is assigned to each possible events

Example

Define F(x) as the frequency of occurrence of the event associated with x

Four dice are tossed 100 times



Random Variables

Example

- X=life length of a transistor={x|0≤x ≤∞}
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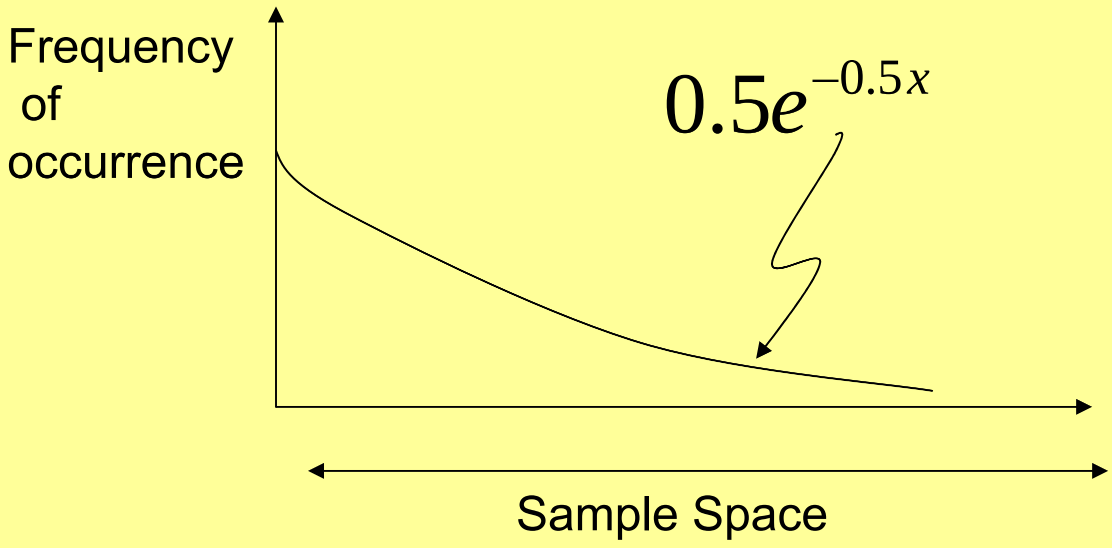
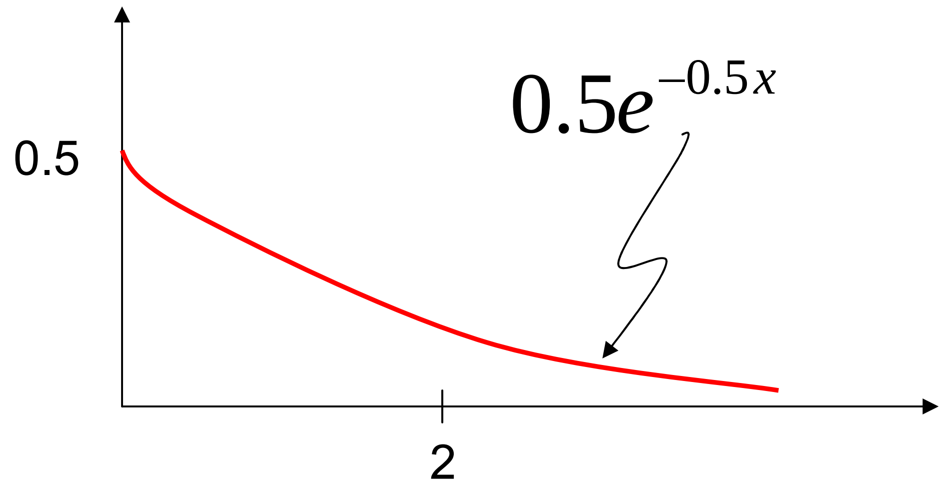


Continuous Random Variable

- A random variable is continuous if it takes on the infinite number of possible values associated with intervals of real numbers



Probability Functions

Probability Density Function

$$p(x)$$

Properties

$$\mathrm{p(x)} \geq 0$$

$$p(-\infty) = 0$$

$$p(\infty) = 0$$

$$p(a \leq x \leq b) = P(b) - P(a)$$

Cumulative Distribution Function

$$P(x) = p(X \leq x)$$

Properties

$$0 \leq P(x) \leq 1$$

$$P(-\infty) = 0$$

$$P(\infty) = 1$$

$$\frac{dP(x)}{dx} \geq 0$$

$$P(x) = \int_{-\infty}^{x} p(x)\, dx$$

Mean and Variance

Random Variables

Example:

4 dice are tossed

x = # of appearance of ‘5’ on the top of four dice toss.

There are four possible values that x can take 0,1,2,3 and 4

X is a random variable. It is a number that is assigned to each possible events

Example

x	F(x)	xF(x)	$x^2F(x)$
0	48	0	0
1	39	39	39
2	11	22	44
3	2	6	18
4	0	0	0
Sums	100	67	101

Mean and Variance

$$Mean = E\{x\} = \bar{x} = \frac{\sum xF(x)}{\sum F(x)}$$

$$Variance = E\{x^2\} = \frac{\sum (x-\bar{x})^2 F(x)}{\sum F(x)} = \lambda^2$$

$$\text{standard deviation} = \lambda$$

$$\text{Third Central Moment} = \frac{\sum (x-\bar{x})^3 F(x)}{\sum F(x)}$$

Mean and Variance

$$\text{Mean} = E\{x\} = \bar{x} = \int_{-\infty}^{\infty} x\, p(x)\, dx$$

$$E\{x^2\} = \int_{-\infty}^{\infty} x^2\, p(x)\, dx$$

$$\text{Variance } V\{x\} = \sigma_x^2 = E\left\{\left(x - E\{x\}\right)^2\right\} = E\{x^2\} - \left(E\{x\}\right)^2$$

$$\text{Properties of E}\{.\}$$

$$E\{aX\} = a\,E\{X\}$$

$$E\{aX + b\} = aE\{X\} + b$$

$$V\{aX + b\} = a^2 V(X)$$

Mean and Variance

Example

x is a random variable with $p(z) = \begin{cases} e^{-z} & z > 0 \\ 0 & z < 0 \end{cases}$

$$E\{x\} = \int_{-\infty}^{\infty} x\, p(x)\, dx = \int_{0}^{\infty} x\, e^{-x}\, dx = 1$$

$$E\{x^2\} = \int_{-\infty}^{\infty} x^2\, p(x)\, dx = \int_{0}^{\infty} x^2\, e^{-x}\, dx = 2$$

Variance $V\{x\} = E\{x^2\} - (E\{x\})^2 = 2 - 1 = 1$

Reading Material

- Chapter 1 J.A. Borrie. Stochastic Systems For Engineers
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