SE301:Numerical Methods
Topic 9
Partial Differential Equations

Dr. Samir Al-Amer
Term 071

Lect 27: Partial Differential Equations

- Partial Differential Equations (PDE)
- What is a PDE
- Examples of Important PDE
- Classification of PDE

Partial Differential Equations

A partial differential equation (**PDE**) is an equation that involves an unknown function and its partial derivatives.

Examples:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$$

PDE involves two or more independent variables

(in the example x and t are independent variables)

Notation

$$u_{xx} = \frac{\partial^2 u(x,t)}{\partial x^2}$$

$$u_{xt} = \frac{\partial^2 u(x,t)}{\partial x \, \partial t}$$

order of the PDE = order of the highest order derivative

Linear PDE

Classification

A PDE is linear if it is linear in the unknown function and its derivatives

Example of linear PDE :

$$2\,u_{xx} + 1\,u_{xt} + 3\,u_{tt} + 4\,u_x + \cos(2t) = 0$$

$$2\,u_{xx} + -3\,u_t + 4\,u_x = 0$$

Examples of Nonlinear PDE

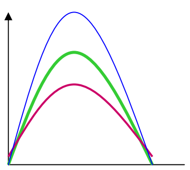
$$2\,u_{xx} + \underline{(u_{xt})^2} + 3\,u_{tt} = 0$$

$$\underline{\sqrt{u_{xx}}} + 2\,u_{xt} + 3u_t = 0$$

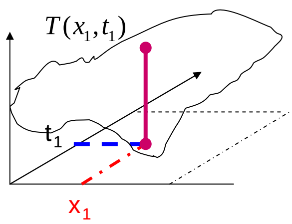
$$2\,u_{xx} + \underline{2\,u_{xt}\,u_t} + 3u_t = 0$$

Representing the solution of PDE (two independent variables)

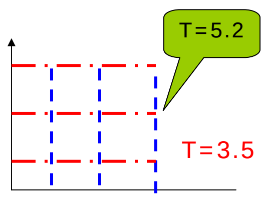
- Three main ways to represent the solution



Different curves are used for different values of one of the independent variable

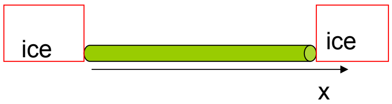


Three dimensional plot of the function T(x,t)



The axis represent the independent variables. The value of the function is displayed at grid points

Heat Equation

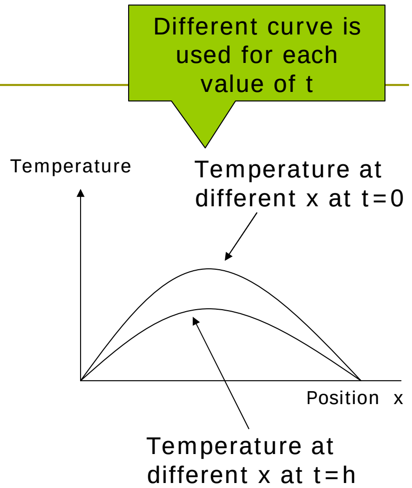


Thin metal rod insulated everywhere except at the edges. At t =0 the rod is placed in ice

$$\frac{\partial^2\, T(x,t)}{\partial x^2} - \frac{\partial\, T(x,t)}{\partial t} = 0$$

$$T(0,t) = T(1,t) = 0$$

$$T(x,o) = \sin(\pi\, x)$$

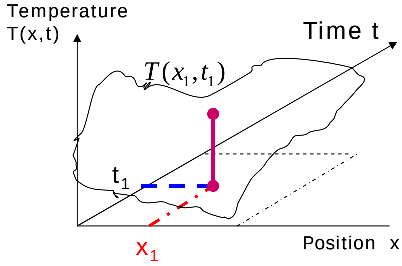


Heat Equation

$$\frac{\partial^2\, T(x,t)}{\partial x^2} - \frac{\partial\, T(x,t)}{\partial t} = 0$$

$$T(0,t) = T(1,t) = 0$$

$$T(x,0) = \sin(\pi\, x)$$



Linear Second Order PDE

Classification

A second order linear PDE (2 - indepent variables)

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D = 0,$$

D is a function of x, y, u_x, u_y

is classfied based on $(B^2 - 4AC)$ as follows:

$B^2 - 4AC = 0$	Parabolic
$B^2 - 4AC > 0$	Hyperbolic
$B^2 - 4AC < 0$	Elliptic

Linear Second Order PDE

Examples (Classification)

Heat Equation $\quad k\dfrac{\partial^2\,T(x,t)}{\partial x^2} - \dfrac{\partial\,T(x,t)}{\partial t} = 0$

$A = k, B = 0, C = 0 \Rightarrow B^2 - 4AC = 0$

$\Rightarrow$ Heat Equation *is Parabolic*

Wave Equation $\quad \dfrac{\partial^2\,u(x,t)}{\partial x^2} - \dfrac{\partial^2\,u(x,t)}{\partial t^2} = 0$

$A = 1, B = 0, C = -1 \Rightarrow B^2 - 4AC > 0$

$\Rightarrow$ Wave Equation *is Hyperbolic*

Classification of PDE

Linear Second order PDE are important set of equations that are used to model many systems in many different fields of science and engineering.

Classification is important because

- Each category relates to specific engineering problems
- Different approaches are used to solve these categories

Examples of PDE

PDE are used to model many systems in many different fields of science and engineering.

Important Examples:

- Wave Equation
- Heat Equation
- Laplace Equation
- Biharmonic Equation

Heat Equation

$$\frac{\partial^2 u(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u(x,y,z,t)}{\partial y^2} + \frac{\partial^2 u(x,y,z,t)}{\partial z^2} = \frac{\partial u(x,y,z,t)}{\partial t}$$

The function *u(x,y,z,t)* is used to represent the temperature at time *t* in a physical body at a point with coordinates *(x,y,z)* .

Simpler Heat Equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$$

u(x,t) is used to represent the temperature at time *t* at the point *x* of the thin rod.

Wave Equation

$$\frac{\partial^2 u(x,y,z,t)}{\partial x^2}+\frac{\partial^2 u(x,y,z,t)}{\partial y^2}+\frac{\partial^2 u(x,y,z,t)}{\partial z^2}=\frac{\partial^2 u(x,y,z,t)}{\partial t^2}$$

The function *u(x,y,z,t)* is used to represent the displacement at time t of a particle whose position at rest is *(x,y,z)* .

Used to model movement of 3D elastic body

Laplace Equation

$$\frac{\partial^2 u(x,y,z,t)}{\partial x^2}+\frac{\partial^2 u(x,y,z,t)}{\partial y^2}+\frac{\partial^2 u(x,y,z,t)}{\partial z^2}=0$$

Used to describe the steady state distribution of heat in a body.

Also used to describe the steady state distribution of electrical charge in a body.

Biharmonic Equation

$$\frac{\partial^4 u(x,y,t)}{\partial x^4} + 2\frac{\partial^4 u(x,y,t)}{\partial x^2\,\partial y^2} + \frac{\partial^4 u(x,y,t)}{\partial y^4} = 0$$

Used in the study of elastic stress.

Boundary conditions for PDE

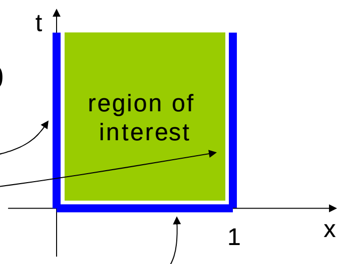
- To uniquely specify a solution to the PDE, a set of boundary conditions are needed.
- Both regular and irregular boundaries are possible

Heat Equation $\dfrac{\partial^2 u(x,t)}{\partial x^2} - \dfrac{\partial u(x,t)}{\partial t} = 0$

$u(0,t) = 0$

$u(1,t) = 0$

$u(x,0) = \sin(\pi x)$



The solution Methods for PDE

- Analytic solutions are possible for simple and special (idealized) cases only.
- To make use of the nature of the equations, different methods are used to solve different classes of PDE.
- The methods discussed here are based on finite difference technique

Elliptic Equations

- Elliptic Equations
- Laplace Equation
- Solution

Elliptic Equations

A second order linear PDE (2 - indepent variables x , y)

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D = 0,$$

where D is a function of x, y, u_x, u_y

is Elliptic if $\boxed{B^2 - 4AC < 0}$

Laplace Equation

Laplace equation appears is several engineering problems such as

- Studying the steady state distribution of heat in a body
- Studying the steady state distribution of electrical in a body

$$\frac{\partial^2\, T(x,y)}{\partial x^2} + \frac{\partial^2\, T(x,y)}{\partial y^2} = f(x,y)$$

T : steady state temperature at point (x, y)

f(x,y): heat source (or heat sink)

Laplace Equation

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = f(x,y)$$

$$A = 1, B = 0, C = 1$$

$$B^2 - 4AC = -4 \quad < 0 \quad Elliptic$$

- Temperature is function of the position (x and y)
- When no heat source is available ☾ f(x,y)=0

Solution Technique

- A grid is used to divide region of interest
- Since the PDE is satisfied at each point in the area, it must be satisfied at each point of the grid.
- A finite difference approximation is obtained at each grid point.

$$\frac{\partial^2 T(x,y)}{\partial x^2} \approx \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2}, \quad \frac{\partial^2 T(x,y)}{\partial y^2} \approx \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}$$

Solution Technique

$$\frac{\partial^2 T(x,y)}{\partial x^2} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2},$$

$$\frac{\partial^2 T(x,y)}{\partial y^2} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}$$

$$\Rightarrow \frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0$$

is approximated by

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0$$

Solution Technique

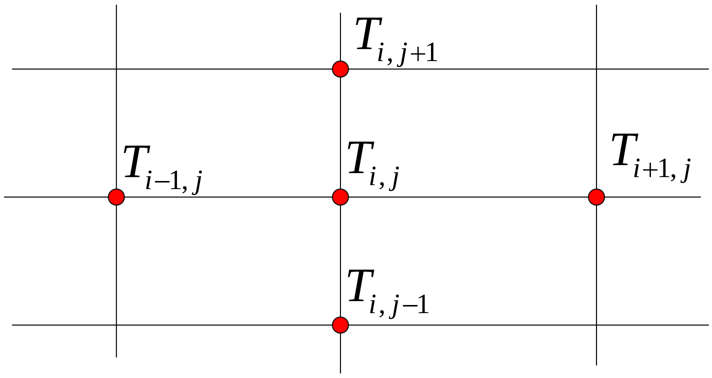
$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0$$

$(Laplacian\ Difference Equation)$

$assume\ \Delta x = \Delta y = h$

$$\Rightarrow T_{i+1,j} - 4T_{i,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} = 0$$

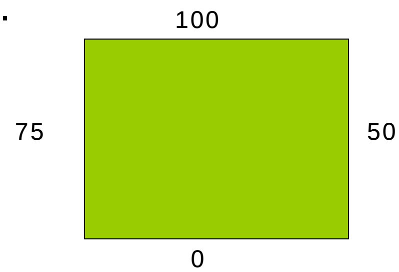
Solution Technique



$$\boxed{T_{i+1,j} - 4T_{i,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} = 0}$$

Example

It is required to determine the steady state temperature at all points of a heated sheet of metal. The edges of the sheet are kept at constant temperature 100,50, 0 and 75 degrees.

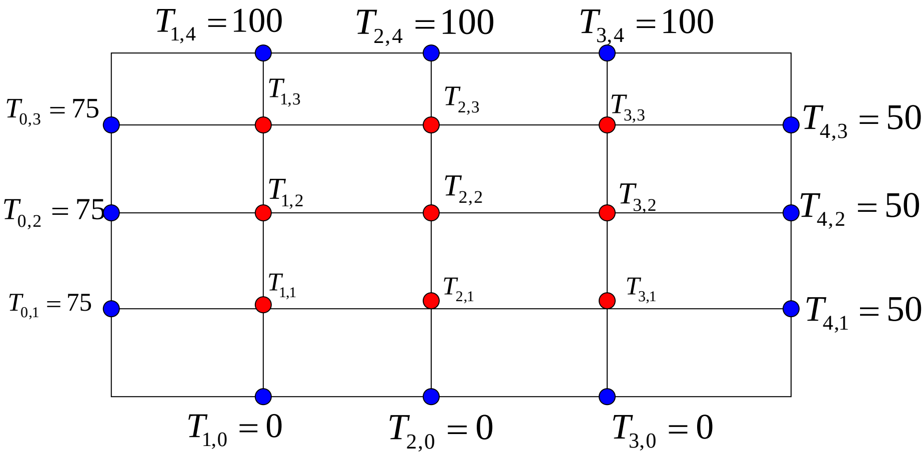


The sheet is divided by 5X5 grids

Known

To be determined

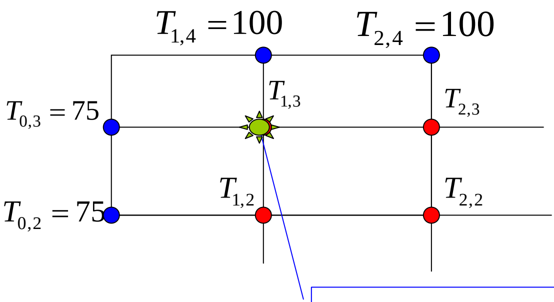
Example



Known

To be determined

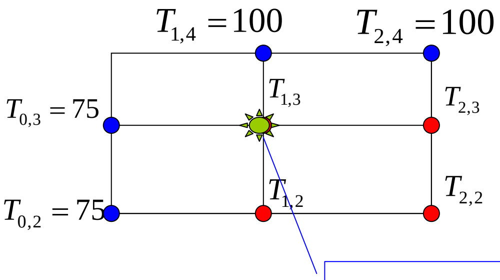
First equation



$$T_{0,3}+T_{1,4}+T_{1,2}+T_{2,3}+-4T_{1,3}=0$$

$$75+100+T_{1,2}+T_{2,3}+-4T_{1,3}=0$$

Example

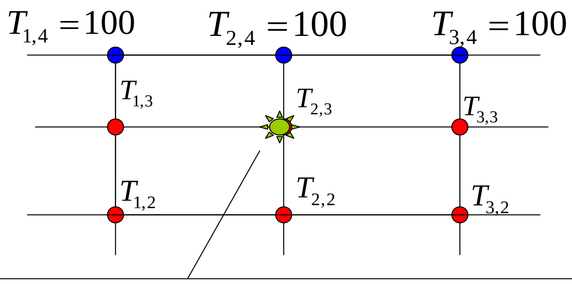


$$T_{0,3} + T_{1,4} + T_{1,2} + T_{2,3} - 4T_{1,3} = 0$$
$$75 + 100 + T_{1,2} + T_{2,3} - 4T_{1,3} = 0$$

Another Equation

Known

To be determined



$$T_{1,3} + T_{2,4} + T_{3,3} + T_{2,2} - 4T_{2,3} = 0$$
$$T_{1,3} + 100 + T_{3,3} + T_{2,2} - 4T_{2,3} = 0$$

Solution

The rest of the equations

$$
\begin{pmatrix}
4 & -1 & 0 & -1 & & & & & \\
-1 & 4 & -1 & 0 & -1 & & & & \\
0 & -1 & 4 & 0 & 0 & -1 & & & \\
-1 & 0 & 0 & 4 & -1 & 0 & -1 & & \\
 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & \\
 & & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\
 & & & -1 & 0 & 0 & 4 & -1 & 0 \\
 & & & & -1 & 0 & -1 & 4 & -1 \\
 & & & & & -1 & 0 & -1 & 4
\end{pmatrix}
\begin{pmatrix} T_{11} \\ T_{21} \\ T_{31} \\ T_{12} \\ T_{22} \\ T_{32} \\ T_{13} \\ T_{23} \\ T_{33} \end{pmatrix}
=
\begin{pmatrix} 75 \\ 0 \\ 50 \\ 75 \\ 0 \\ 50 \\ 175 \\ 100 \\ 150 \end{pmatrix}
$$

Convergence and stability of solution

- Convergence

  The solutions converge means that the solution obtained using finite difference method approaches the true solution as the steps Δx *and* Δt approaches zero.

- Stability:

  An algorithm is stable if the errors at each stage of the computation are not magnified as the computation progresses.

Parabolic Equations

- Parabolic Equations
- Heat Conduction Equation
- Explicit Method
- Implicit Method
- Cranks Nicolson Method

Parabolic Equations

A second order linear PDE (2-indepent variables x , y)

$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D = 0,$

where D is a function of x, y, u_x, u_y

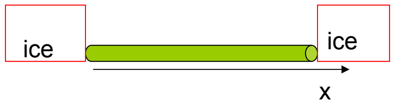
is parabolic if $\boxed{B^2 - 4AC = 0}$

Parabolic Problems

Heat Equation $$\frac{\partial^2 T(x,t)}{\partial x^2} - \frac{\partial T(x,t)}{\partial t} = 0$$

$$\boxed{T(0,t) = T(1,t) = 0}$$

$$\boxed{T(x,o) = \sin(\pi x)}$$



* Parabolic problem $(B^2 - 4AC = 0)$
* Boundary conditions are needed to uniquely specify a solution

First order Partial derivative Finite Difference

Central difference Method	Forward difference Method	Backward difference Method
$\frac{\partial T}{\partial x} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{2\ \Delta x}$	$\frac{\partial T}{\partial x} = \frac{T_{i+1,j} - T_{i,j}}{\Delta x},$	$\frac{\partial T}{\partial x} = \frac{T_{i,j} - T_{i-1,j}}{\Delta x}$

Finite Difference Methods

Replace the derivatives by finite difference formula [T is function of x and t]

Central Difference Formulas :

$$\frac{\partial T(x,t)}{\partial x} \approx \frac{T_{i+1,j} - T_{i-1,j}}{2\,\Delta x}$$

$$\frac{\partial^2\ T(x,t)}{\partial x^2} \approx \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2}$$

$$\frac{\partial^2\ u(x,t)}{\partial t^2} \approx \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta t)^2}$$

Forward Difference Formula :

$$\frac{\partial T(x,t)}{\partial x} \approx \frac{T_{i+1,j} - T_{i,j}}{\Delta x}, \quad \frac{\partial T(x,t)}{\partial t} \approx \frac{T_{i,j+1} - T_{i,j}}{\Delta t}$$

Finite Difference Methods

New Notation

Central Difference Formulas :

$$\frac{\partial T(x,t)}{\partial x} \approx \frac{T_{i+1}^{l} - T_{i-1}^{l}}{2\,\Delta x}$$

$$\frac{\partial^2\ T(x,t)}{\partial x^2} \approx \frac{T_{i+1}^{l} - 2T_{i}^{l} + T_{i-1}^{l}}{(\Delta x)^2}$$

$$\frac{\partial^2\ T(x,t)}{\partial t^2} \approx \frac{T_{i}^{l+1} - 2T_{i}^{l} + T_{i}^{l-1}}{(\Delta t)^2}$$

Superscript for t-axis
And
Subscript for x-axis
$T_i^{l-1} = T(x, t-\Delta t)$

Forward Difference Formula :

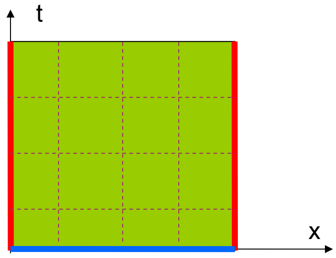
$$\frac{\partial T(x,t)}{\partial x} \approx \frac{T_{i+1}^{l} - T_{i}^{l}}{\Delta x}, \quad \frac{\partial T(x,t)}{\partial t} \approx \frac{T_{i}^{l+1} - T_{i}^{l}}{\Delta t}$$

Solution of the PDE

Heat Equation $\frac{\partial^2 T(x,t)}{\partial x^2} - \frac{\partial T(x,t)}{\partial t} = 0$

$\boxed{T(0,t) = T(1,t) = 0}$

$\boxed{T(x,o) = \sin(\pi x)}$



Solution means

Determine the value of T(x, t) at the grid points

Solution of the Heat Equation

Two solutions to the Parabolic Equation (Heat Equation) will be presented

1. Explicit Method:

    Simple, Stability Problems

2. Crank-Nicolson Method:

    involves solution of Tridiagonal system of equations, stable.

Explicit Method

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$\frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} = \frac{u(x,t+k) - u(x,t)}{k}$$

$$\frac{1}{h^2}\big(u(x+h,t) - 2u(x,t) + u(x-h,t)\big) - \frac{1}{k}\big(u(x,t+k) + u(x,t)\big) = 0$$

$$Define \quad \sigma = \frac{k}{h^2}$$

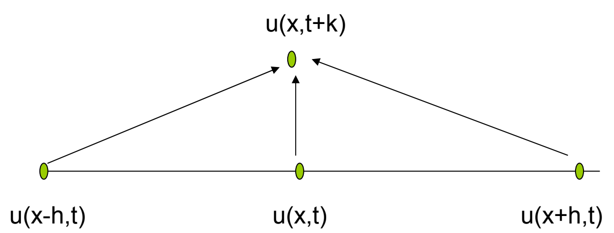
$$u(x,t+k) = \sigma\ u(x+h,t) + (1-2\sigma)\ u(x,t) + \sigma\ u(x-h,t)$$

Explicit Method

How do we compute

$$u(x,t+k) = \sigma\ u(x+h,t) + (1-2\sigma)\ u(x,t) + \sigma\ u(x-h,t)$$

means

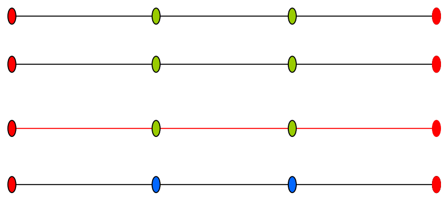


Explicit Method

How do we compute

$$u(x,t+k) = \sigma\ u(x+h,t) + (1-2\sigma)\ u(x,t) + \sigma\ u(x-h,t)$$

means



Explicit Method

$u(x,t+k)$ can be computed directly using

$$u(x,t+k) = \sigma\ u(x+h,t) + (1-2\sigma)\ u(x,t) + \sigma\ u(x-h,t)$$

can be unstable (errors are magnified)

To guarantee stability $(1-2\sigma) > 0 \quad or \quad k \le \frac{h^2}{2}$

this make it slow

Crank-Nicolson Method

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$\frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} = \frac{u(x,t) - u(x,t-k)}{k}$$

$$\frac{1}{h^2}\big(u(x+h,t) - 2u(x,t) + u(x-h,t)\big) - \frac{1}{k}\big(u(x,t) + u(x,t-k)\big) = 0$$

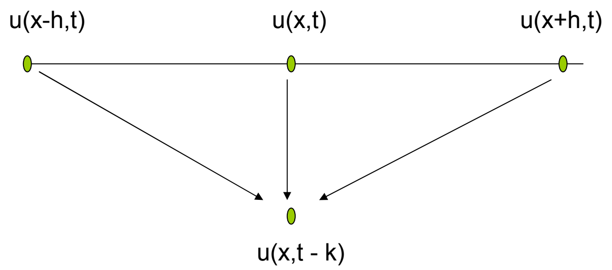
$$Define \quad s = \frac{h^2}{k}, \quad r = 2 + s$$

$$s\ u(x,t-k) = -\ u(x-h,t) + r\ u(x,t) - u(x+h,t)$$

Explicit Method
How do we compute

$$su(x,t-k) = u(x+h,t) + r\ u(x,t) + \ u(x-h,t)$$

means



Crank-Nicolson Method

The equation

$$s\,u(x,t-k) = -\,u(x-h,t) + r\,u(x,t) - u(x+h,t)$$

can be expressed as a Tridiagonal system of equations

$$\begin{bmatrix} r & -1 & & \\ -1 & r & -1 & \\ & -1 & r & -1 \\ & & -1 & r \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Crank-Nicolson Method

The method involves solving a Tridiagonal system of linear equations

The method is stable (No magnification of error)

$\rightarrow$ We can use larger h, k (compared to the Explicit Method)

Outlines

- Examples
  - Explicit method to solve Parabolic PDE
  - Cranks-Nicholson Method

Heat Equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$\boxed{u(0,t) = u(1,t) = 0}$$

$$\boxed{u(x,o) = \sin(\pi x)}$$

* Parabolic problem $(B^2 - 4AC = 0)$

* Auxiliary conditions are needed to uniquely specify a solution

Example 1

Solve the PDE

$$\frac{\partial^2 u(x,t)}{\partial^2 x} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = \sin(\pi x)$$

$Use \quad h = 0.25, \quad k = 0.25$ to find u(t, x) for $x \in [0,1], t \in [0,1]$

$$\sigma = \frac{k}{h^2} = 4$$

Example 1 (cont.)

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$$

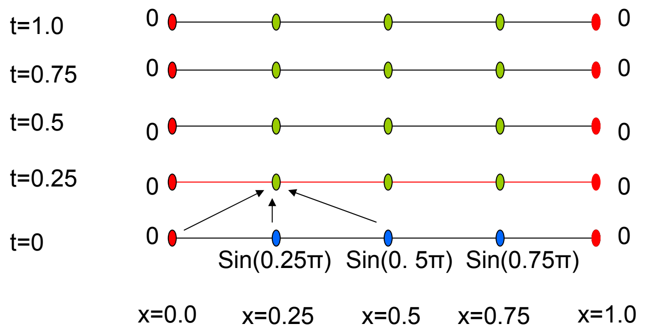
$$\frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} - \frac{u(x,t+k) - u(x,t)}{k} = 0$$

$$16\left(u(x+h,t) - 2u(x,t) + u(x-h,t)\right) - 4\left(u(x,t+k) + u(x,t)\right) = 0$$

$$u(x,t+k) = 4\ u(x+h,t) - 7\ u(x,t) + 4\ u(x-h,t)$$

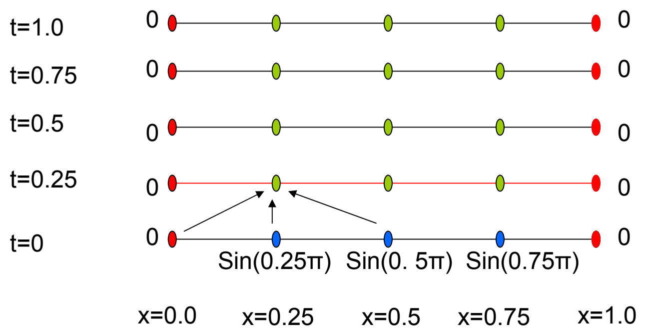
Example 1

$$u(x,t+k) = 4\,u(x+h,t) - 7\,u(x,t) + 4\,u(x-h,t)$$



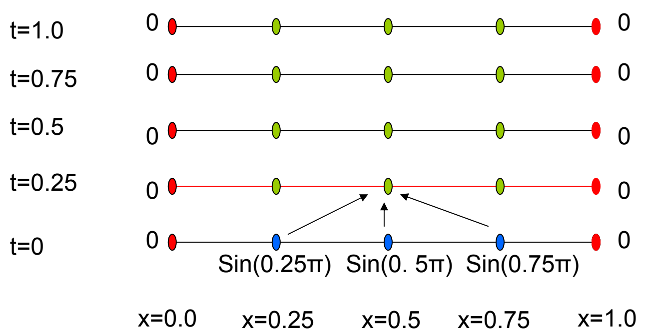
Example 1

$$u(0.25,0.25) = 4\,u(.5,0) - 7\,u(.25,0) + 4\,u(0,0)$$
$$= 4\sin(\pi/2) - 7\sin(\pi/4) + 0 = -0.9497$$



Example 1

$$u(0.5,0.25) = 4\,u(0.75,0) - 7\,u(0.5,0) + 4\,u(0.25,0)$$
$$= 4\sin(3\pi/4) - 7\sin(\pi/2) + 4\sin(\pi/2) = -0.1716$$



Remarks on Example 1

The obtained results are probably not accurate because $1-2\sigma = -7$

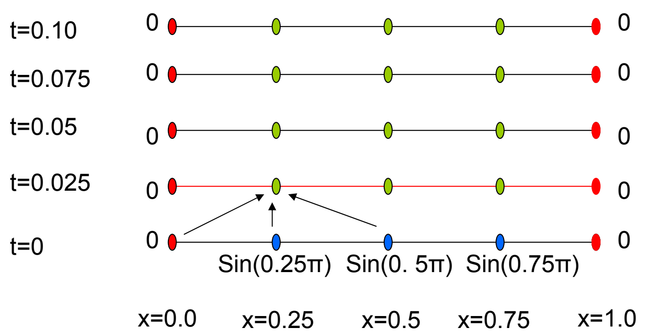
For accurate results $1-2\sigma > 0$

One need to select $k < 0.03125$

Let $k = 0.025$

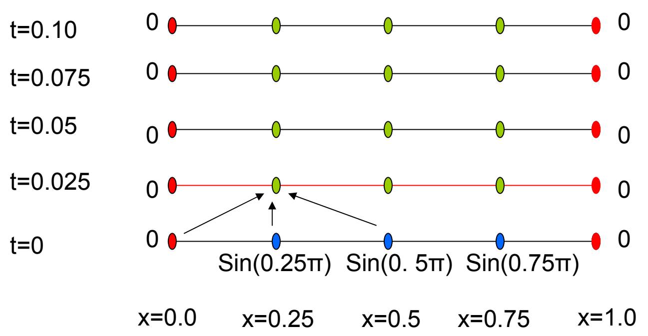
Example 1

$$u(x,t+k)=0.4\,u(x+h,t)+0.2\,u(x,t)+0.4\,u(x-h,t)$$



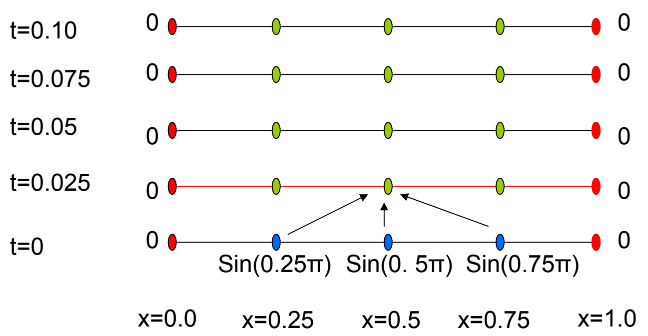
Example 1

$$u(0.25,0.025)=0.4\,u(0.5,0)+0.2\,u(0.25,0)+0.4\,u(0,0)$$
$$=0.4\sin(\pi/2)+.2\sin(\pi/4)+0=0.5414$$



Example 1

$$u(0.5,0.025) = 0.4\ u(0.75,0) + 0.2\ u(0.5,0) + 0.4\ u(0.25,0)$$
$$= 0.4\sin(3\pi/4) + .2\sin(\pi/2) + 0.4\sin(\pi/4) = 0.7657$$



Example 2

Solve the PDE

$$\frac{\partial^2 u(x,t)}{\partial^2 x} - \frac{\partial u(x,t)}{\partial t} = 0$$
$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = \sin(\pi\, x)$$

Solve using Crank - Nicolson method

$Use \quad h = 0.25, \quad k = 0.25$ to find u(t, x) for $x \in [0,1], t \in [0,1]$

Example 2

Crank-Nicolson Method

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$\frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} = \frac{u(x,t) - u(x,t-k)}{k}$$

$$16\big(u(x+h,t) - 2u(x,t) + u(x-h,t)\big) - 4\big(u(x,t) - u(x,t-k)\big) = 0$$

$$Define \quad s = \frac{h^2}{k} = 0.25, \quad r = 2 + s = 2.5$$

$$0.25u(x,t-k) = -u(x-h,t) + 2.5\, u(x,t) - (x+h,t)$$

$$u(x,t-k) = -4u(x-h,t) + 10\, u(x,t) - 4(x+h,t)$$

Example 2

Crank-Nicolson Method

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$\frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} = \frac{u(x,t) - u(x,t-k)}{k}$$

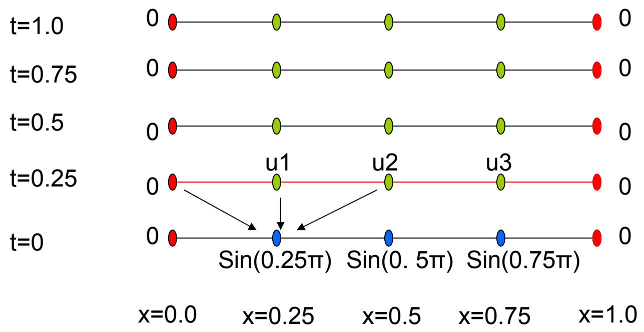
$$16\big(u(x+h,t) - 2u(x,t) + u(x-h,t)\big) - 4\big(u(x,t) - u(x,t-k)\big) = 0$$

$$Define\ s = \frac{h^2}{k} = 0.25, \quad r = 2 + s = 2.25$$

$$0.25u(x,t-k) = -u(x-h,t) + 2.25\, u(x,t) - (x+h,t)$$

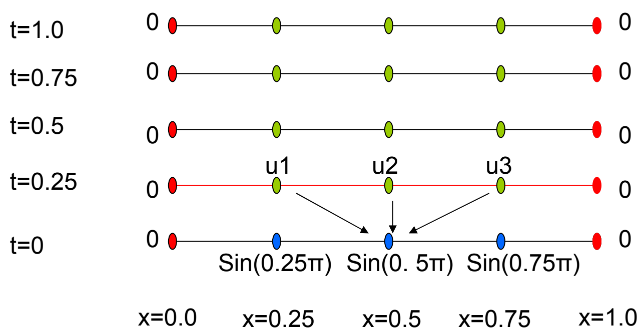
Example 2

$$0.25u(0.0,0.25) = -\ u(0,0.25) + 2.25\ u(0.25,0.25) - u(0.5,0.25)$$
$$0.25\sin(0.25\pi) = 0 + 2.25u_1 - u_2$$



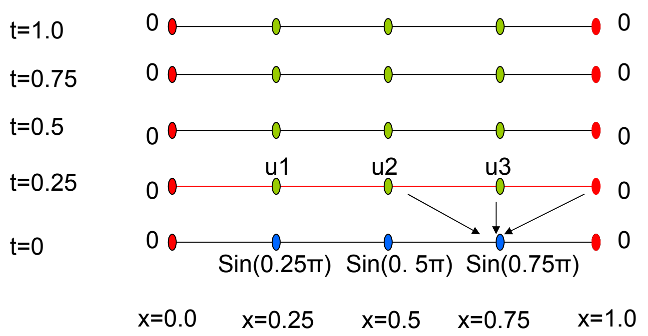
Example 2

$$0.25u(0,0.5) = -\ u(0.25,0.25) + 2.25\ u(.5,0.25) - u(0.75,0.25)$$
$$0.25\sin(0.5\pi) = -u_1 + 2.25u_2 - u_3$$



Example 2

$$0.25u(0.75,0) = -\ u(0.5,0.25) + 2.25\ u(.75,0.25) - \ u(1,0.25)$$
$$0.25\sin(0.75\pi) = -u_2 + 2.25u_3 - 0$$



Example 2
Crank-Nicolson Method

The solution of the PDE is converted to solution of the following tridiagonal system

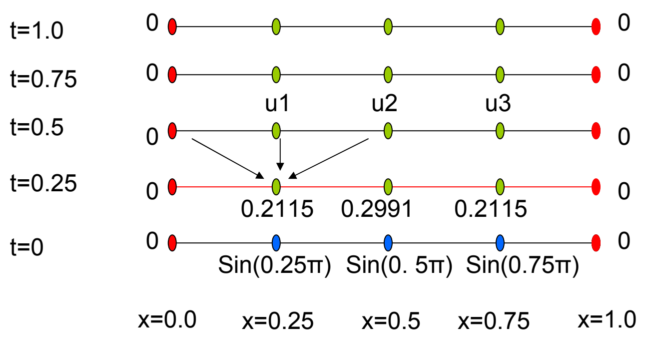
$$\begin{bmatrix} 2.25 & -1 & \\ -1 & 2.25 & -1 \\ & -1 & 2.25 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.25\sin(0.25\pi) \\ 0.25\sin(0.5\pi) \\ 0.25\sin(0.75\pi) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.21152 \\ 0.2912 \\ 0.21151 \end{bmatrix}$$

Example 2
Second Row

$$0.25u(0.25,0.25) = -\,u(0,0.5) + 2.25\,u(0.25,0.5) - u(0.5,0.5)$$

$$0.2115 = 0 + 2.25u_1 - u_2$$



Example 2

The process is continued until the values of u(x,t) on the desired grid are computed.

Remarks

The explicit metod:

- one need to select small k to ensure **stability**
- Computation per point is very simple but many points are needed.

Cranks Nicolson

- Requires solution of **Tridiagonal** system
- Stable (larger k can be used).