SE301:Numerical Methods
Topic 5
Interpolation

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(Term 063)

Read Chapter 18, Sections 1-5

SE301:Numerical Methods Lecture 14: Introduction to Interpolation

- Introduction
- Interpolation problem
- Existence and uniqueness
- Linear and Quadratic interpolation
- Newton's Divided Difference method
- Properties of Divided Differences

Introduction

Interpolation was used for long time to provide an estimate of a tabulated function at values that are not available in the table.

What is sin (0.15)?

x	sin(x)
0	0.0000
0.1	0.0998
0.2	0.1987
0.3	0.2955
0.4	0.3894

Using **Linear Interpolation** sin (0.15) ≈ **0.1493**

True value(4 decimal digits) sin (0.15)= **0.1494**

The interpolation Problem

Given a set of n+1 points,

$$\left(x_0, f(x_0)\right), \left(x_1, f(x_1)\right), \ldots., \left(x_n, f(x_n)\right)$$

find an n[th] order polynomial $f_n(x)$ that passes through all points

$$f_n(x_i) = f(x_i) \quad for \ i = 0,1,2,\ldots,n$$

Example

An experiment is used to determine the viscosity of water as a function of temperature. The following table is generated

Problem: Estimate the viscosity when the temperature is 8 degrees.

Temperature (degree)	Viscosity
0	1.792
5	1.519
10	1.308
15	1.140

Interpolation Problem

Find a polynomial that fit the data points exactly.

$$V(T) = \sum_{k=0}^{n} a_k \, T^k$$

$$V_i = V(T_i)$$

V : viscosity

T : Temperature

a_k : polynomial coefficients

Linear Interpolation V(T)= 1.73 − 0.0422 T

V(8)= 1.3924

Existence and Uniqueness

Given a set of n+1 points

$$\left(x_0, f(x_0)\right), \left(x_1, f(x_1)\right), \;...., \left(x_n, f(x_n)\right)$$

Assumption $x_0, x_1, \ldots, x_n$ are distinct

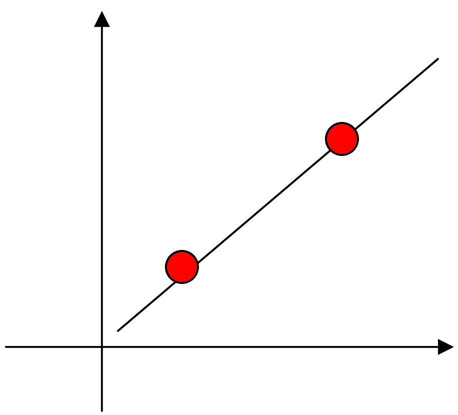
Theorem:

There is a unique polynomial ***$f_n(x)$*** of order ≤ n such that

$$f_n(x_i) = f(x_i) \qquad for \quad i = 0,1,\ldots,n$$

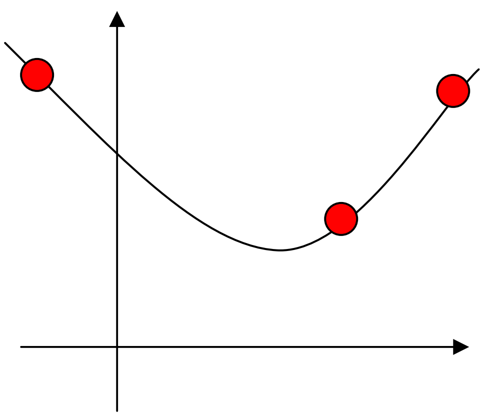
Examples of Polynomial Interpolation

Linear Interpolation



- Given any two points, there is one polynomial of order ≤ 1 that passes through the two points.

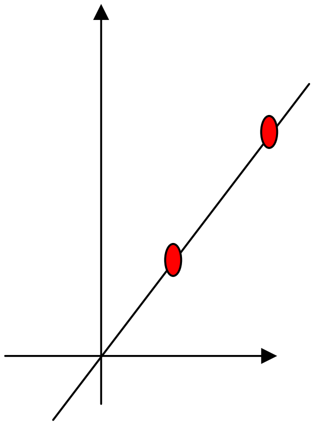
Quadratic Interpolation



Given any three points there is one polynomial of order ≤ 2 that passes through the three points.

Linear Interpolation

Given any two points, $\left(x_0, f(x_0)\right), \left(x_1, f(x_1)\right)$



The line that interpolates the two points is

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}\left(x - x_0\right)$$

Example :

find a polynomial that interpolates (1,2) and (2,4)

$$f_1(x) = 2 + \frac{4-2}{2-1}\left(x-1\right) = 2x$$

Quadratic Interpolation

- Given any three points, $(x_0, f(x_0)), (x_1, f(x_1))$ and $(x_2, f(x_2))$
- The polynomial that interpolates the three points is

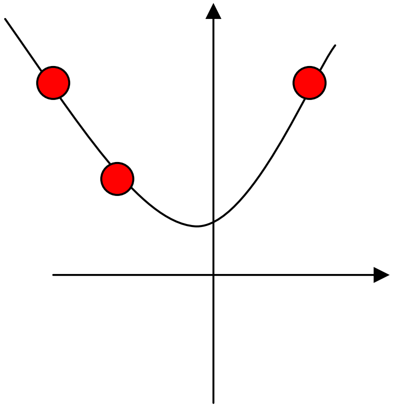
$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f(x_0)$$

$$b_1 = f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_0, x_1, x_2] = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



General n^th order Interpolation

Given any n+1 points, $\left(x_0, f(x_0)\right), \left(x_1, f(x_1)\right), \ldots, \left(x_n, f(x_n)\right)$

The polynomial that interpolates all points is

$$f_n(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + \ldots + b_n(x - x_0)\ldots(x - x_{n-1})$$

$$b_0 = f(x_0)$$

$$b_1 = f[x_0, x_1]$$

....

$$b_n = f[x_0, x_1, \ldots, x_n]$$

Divided Differences

$f[x_k] = f(x_k)$ zeroth order DD

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$ first order DD

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$ Second order DD

....

$$f[x_0, x_1, \ldots, x_k] = \frac{f[x_1, x_2, \ldots, x_k] - f[x_0, x_1, \ldots, x_{k-1}]}{x_k - x_0}$$

Divided Difference Table

x	F[ ]	F[,]	F[, ,]	F[, , ,]
x_0	F[x0]	$F[x_0,x_1]$	$F[x_0,x_{1,}x_2]$	$F[x_0,x_1,x_2,x_3]$
x_1	$F[x_1]$	$F[x_1,x_2]$	$F[x_1,x_2,x_3]$	
x_2	$F[x_2]$	$F[x_2,x_3]$		
x_3	$F[x_3]$			

$$f_n(x) = \sum_{i=0}^{n} \left\{ F[x_0, x_1, \ldots, x_i] \prod_{j=0}^{i-1} \left(x - x_j\right) \right\}$$

Divided Difference Table

x	F[ ]	F[,]	F[, ,]
0	-5	2	-4
1	-3	6	
-1	-15		

x_i	$f(x_i)$
0	-5
1	-3
-1	-15

Entries of the divided difference table are obtained from the data table using simple operations

Divided Difference Table

x	F[ ]	F[,]	F[, ,]
0	-5	2	-4
1	-3	6	
-1	-15		

x_i	$f(x_i)$
0	-5
1	-3
-1	-15

The first two column of the

table are the data columns

Third column : first order differences

Fourth column: Second order differences

Divided Difference Table

x	F[ ]	F[,]	F[, ,]
0	-5	2	-4
1	-3	6	
-1	-15		

$$\frac{-3-(-5)}{1-0} = 2$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

x_i	y_i
0	-5
1	-3
-1	-15

Divided Difference Table

x	F[ ]	F[,]	F[, ,]
0	-5	2	-4
1	-3	6	
-1	-15		

$$\frac{-15-(-3)}{-1-1}=6$$

$$f[x_1,x_2]=\frac{f[x_2]-f[x_1]}{x_2-x_1}$$

x_i	y_i
0	-5
1	-3
-1	-15

Divided Difference Table

x	F[ ]	F[,]	F[, ,]
0	-5	2	-4
1	-3	6	
-1	-15		

$$\frac{6-(2)}{-1-(0)} = -4$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

x_i	y_i
0	-5
1	-3
-1	-15

Divided Difference Table

x	F[ ]	F[,]	F[, ,]
0	-5	2	-4
1	-3	6	
-1	-15		

x_i	y_i
0	-5
1	-3
-1	-15

$$f_2(x) = -5 + 2(x-0) - 4(x-0)(x-1)$$

$$f_2(x) = F[x_0] + F[x_0, x_1]\,(x-x_0) + F[x_0, x_1, x_2]\,(x-x_0)(x-x_1)$$

Two Examples

Obtain the interpolating polynomials for the two examples

x	y
1	0
2	3
3	8

x	y
2	3
1	0
3	8

What do you observe?

Two Examples

x	Y		
1	0	3	1
2	3	5	
3	8		

$$P_2(x) = 0 + 3(x-1) + 1(x-1)(x-2)$$
$$= x^2 - 1$$

x	Y		
2	3	3	1
1	0	4	
3	8		

$$P_2(x) = 3 + 3(x-2) + 1(x-2)(x-1)$$
$$= x^2 - 1$$

Ordering the points should not affect the interpolating polynomial

Properties of divided Difference

Ordering the points should not affect the divided difference

$$f[x_0, x_1, x_2] = f[x_1, x_2, x_0] = f[x_2, x_1, x_0]$$

Example

- Find a polynomial to interpolate the data.

x	f(x)
2	3
4	5
5	1
6	6
7	9

Example

x	f(x)	f[,]	f[, ,]	f[, , ,]	f[, , , ,]
2	3	1	-1.6667	1.5417	-0.6750
4	5	-4	4.5	-1.8333	
5	1	5	-1		
6	6	3			
7	9				

$$f_4 = 3 + 1(x-2) - 1.6667(x-2)(x-4) + 1.5417(x-2)(x-4)(x-5)$$
$$- 0.6750(x-2)(x-4)(x-5)(x-6)$$

Summary

Interpolating Condition : $f(x_i) = f_n(x_i) \quad for \quad i = 0,1,2,\ldots,n$

* The interpolating Polynomial is unique.

* Different methods can be used to obtain it

- Newton Divided Difference [Section 18.1]
- Lagrange Interpolation [Section 18. 2]
- Other methods

Ordering the points should not affect the interpolating polynomial

SE301: Numerical Methods

Lecture 15

Lagrange Interpolation

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The interpolation Problem

Given a set of n+1 points,

$$\left(x_0, f(x_0)\right), \left(x_1, f(x_1)\right), \ldots., \left(x_n, f(x_n)\right)$$

find an n[th] order polynomial $f_n(x)$

that passes through all points

$$f_n(x_i) = f(x_i) \quad for\ i = 0,1,2,\ldots,n$$

Lagrange Interpolation

Problem:

Given

x_i	x_0	x_1	....	x_n
y_i	y_0	y_1	....	y_n

Find the polynomial of least order $f_n(x)$ such that

$$f_n(x_i) = f(x_i) \qquad for \quad i = 0,1,\ldots,n$$

Lagrange Interpolation Formula

$$f_n(x) = \sum_{i=0}^{n} f\left(x_i\right)\ \ell_i(x)$$

$$\ell_i(x) = \prod_{j=0,\, j\neq i}^{n} \frac{\left(x - x_j\right)}{\left(x_i - x_j\right)}$$

Lagrange Interpolation

$\ell_i(x)$ are called the cardinals

The cardinals are n th order polynomials

$$\ell_i(x_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Lagrange Interpolation Example

x	1/3	1/4	1
y	2	-1	7

$$P_2(x) = f(x_0)\ell_0(x) + f(x_1)\ell_1(x) + f(x_2)\ell_2(x)$$

$$\ell_0(x) = \frac{(x-x_1)}{(x_0-x_1)}\frac{(x-x_2)}{(x_0-x_2)} = \frac{(x-1/4)}{(1/3-1/4)}\frac{(x-1)}{(1/3-1)}$$

$$\ell_1(x) = \frac{(x-x_0)}{(x_1-x_0)}\frac{(x-x_2)}{(x_1-x_2)} = \frac{(x-1/3)}{(1/4-1/3)}\frac{(x-1)}{(1/4-1)}$$

$$\ell_2(x) = \frac{(x-x_0)}{(x_2-x_0)}\frac{(x-x_1)}{(x_2-x_1)} = \frac{(x-1/3)}{(1-1/3)}\frac{(x-1/4)}{(1-1/4)}$$

$$P_2(x) = 2\{-18(x-1/4)(x-1)\} - 1\{16(x-1/3)(x-1)\}$$
$$+7\{2(x-1/3)(x-1/4)\}$$

Example

find a polynomial to interpolate

both Newton's interpolation method and Lagrange interpolation method must give the same answer

x	y
0	1
1	3
2	2
3	5
4	4

0	1	2	-3/2	7/6	-5/8
1	3	-1	2	-4/3	
2	2	3	-2		
3	5	-1			
4	4				

Interpolating Polynomial

$$f_4(x) = 1 + 2(x) - \frac{3}{2}x(x-1) + \frac{7}{6}x(x-1)(x-2)$$

$$-\frac{5}{8}x(x-1)(x-2)(x-3)$$

Interpolating Polynomial Using Lagrange Interpolation method

$$f_4(x) = \sum_{i=0}^{4} f(x_i)\ \ell_i = \ell_0 + 3\ell_1 + 2\ell_2 + 5\ell_3 + 4\ell_4$$

$$\ell_0 = \frac{x-1}{0-1}\ \frac{x-2}{0-2}\ \frac{x-3}{0-3}\ \frac{x-4}{0-4}$$

$$\ell_1 = \frac{x-0}{1-0}\ \frac{x-2}{1-2}\ \frac{x-3}{1-3}\ \frac{x-4}{1-4}$$

$$\ell_2 = \frac{x-0}{2-0}\ \frac{x-1}{2-1}\ \frac{x-3}{2-3}\ \frac{x-4}{2-4}$$

$$\ell_3 = \frac{x-0}{3-0}\ \frac{x-1}{3-1}\ \frac{x-2}{3-2}\ \frac{x-4}{3-4}$$

$$\ell_4 = \frac{x-0}{4-0}\ \frac{x-1}{4-1}\ \frac{x-2}{4-2}\ \frac{x-3}{4-3}$$

SE301Numerical Methods

Lecture 16

Inverse interpolation

Error in polynomial interpolation

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Inverse Interpolation

x_i	x_0	x_1	....	x_n
y_i	y_0	y_1	....	y_n

Problem :

Given the table and y_g

Find x_g such that $f(x_g) = y_g$

One approach:

Use polynomial interpolation to obtain $f_n(x)$ to interpolate the data then use Newton's method to find a solution to

$$f_n(x_g) = y_g$$

Inverse Interpolation

Alternative Approach

Inverse interpolation:

1. Exchange the roles of x and y

x_i	x_0	x_1	....	x_n
y_i	y_0	y_1	....	y_n

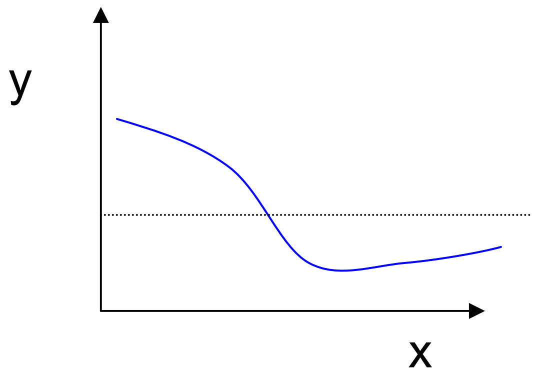
2. Perform polynomial Interpolation on the new table.

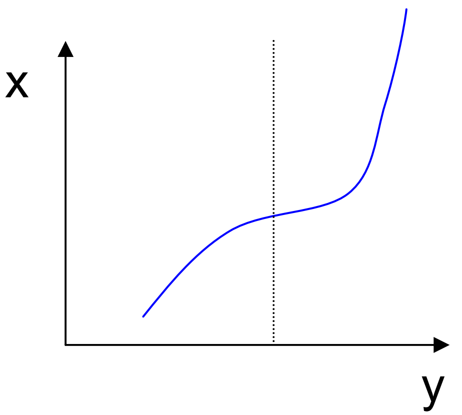
y_i	y_0	y_1	....	y_n
x_i	x_0	x_1	....	x_n

3. Evaluate

$$f_n(y_g)$$

Inverse Interpolation





Inverse Interpolation

Question:

What is the limitation of inverse interpolation

- The original function has an inverse
- $y_1, y_2, \ldots, y_n$ must be distinct.

Inverse Interpolation

Example

x	1	2	3
y	3.2	2.0	1.6

Problem :

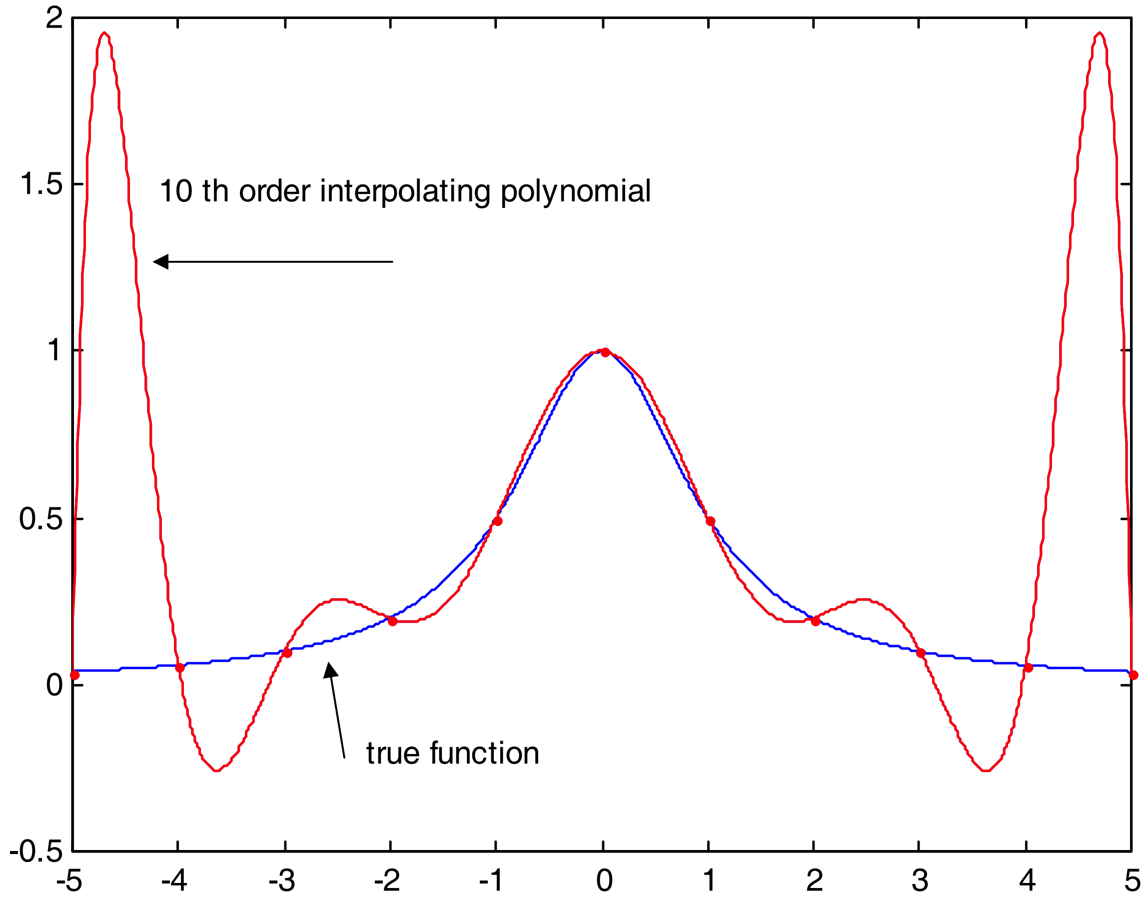
Given the table. Find x_g such that $f(x_g) = 2.5$

3.2	1	-.8333	1.0416
2.0	2	-2.5	
1.6	3		

$$f_2(y) = 1 - 0.8333(y - 3.2) + 1.0416(y - 3.2)(y - 2)$$

$$f_2(2.5) = 1 - 0.8333(-0.7) + 1.0416(-0.7)(0.5) = 1.2187$$

10 th order Polynomial Interpolation



Errors in polynomial Interpolation

- Polynomial interpolation may lead to large error (especially for high order polynomials)

BE CAREFUL

- When an nth order interpolating polynomial is used, the error is related to the (n+1) th order derivative

Errors in polynomial Interpolation

Theorem

Let f(x) be a function such that

$f^{(n+1)}(x)$ is continuous on $[a,b]$, $\left| f^{(n+1)} \right| \le M$

Let P(x) be any polynomial of degree $\le n$

that interpolates f at $n+1$ equally spaced points

in $[a,b]$ (including the end points). Then

$$\left| f(x) - P(x) \right| \le \frac{M}{4(n+1)} \left(\frac{b-a}{n} \right)^{n+1}$$

Example 1

$\mathrm{f(x) = sin(x)}$

We want to use 9 th order polynomial to interpolate f(x)

(using 10 equally spaced points) in the interval [0,1.6875]

$$\left| f^{(n+1)} \right| \leq 1 \quad for\ \ n > 0$$

$$M = 1, n = 9$$

$$\left| \mathrm{f(x) - P(x)} \right| \leq \frac{M}{4(n+1)} \left(\frac{b-a}{n} \right)^{n+1}$$

$$\left| \mathrm{f(x) - P(x)} \right| \leq \frac{1}{4(10)} \left(\frac{1.6875}{9} \right)^{10} = 1.34 \times 10^{-9}$$

Interpolation Error

If $f_n(x)$ is the polynomial of degree n that interpolates the function f(x) at the nodes $x_0, x_1, \ldots, x_n$ then for any x not a node

$$f(x) - f_n(x) = f[x_0, x_1, \ldots, x_n, x] \prod_{i=0}^{n} (x - x_i)$$

Not useful. Why?

Approximation of the Interpolation Error

If $f_n(x)$ is the polynomial of degree n that interpolates the function f(x) at the nodes $x_0, x_1, ..., x_n$

$Let(x_{n+1}, f(x_{n+1}))$ *be* an additional point

The interpolation error is approximated by

$$f(x) - f_n(x) \approx f[x_0, x_1, ..., x_n, x_{n+1}] \prod_{i=0}^{n} (x - x_i)$$

Divided Difference Theorem

If $f^{(n)}$ is continuous on [a, b] and if $x_0, x_1, \ldots, x_n$ are any n + 1 distinct points in [a, b] then for some $\xi \in [a, b]$

$$f[x_0, x_1, \ldots, x_n] = \frac{1}{n!} f^{(n)}(\xi)$$

If $f(x)$ *is* a polynomial of order n then

$$f[x_0, x_1, \ldots, x_i] = 0 \quad \forall i \geq n+1$$

1	4	1	0	0
2	5	1	0	
3	6	1		
4	7			

$$f(x) = 3 + x$$

Summary

- The interpolating polynomial is unique
- Different methods can be used to obtain it
  - Newton's Divided difference
  - Lagrange interpolation
  - others
- Polynomial interpolation can be sensitive to data.
- **BE CAREFUL** when high order polynomials are used

HW

- Chick WebCT for HW problems and due dates