SE 207: Modeling and Simulation

Lesson 3: Review of Complex Analysis

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Term 072

Complex Numbers

- Complex numbers: number of the form

$$z = x + j\ y \qquad j = \sqrt{-1}$$

where *x* and *y* are real numbers and

- x: real part of z; x = $Re\{z\}$
- y: imaginary part of z; y = $Im\{z\}$

Complex Numbers

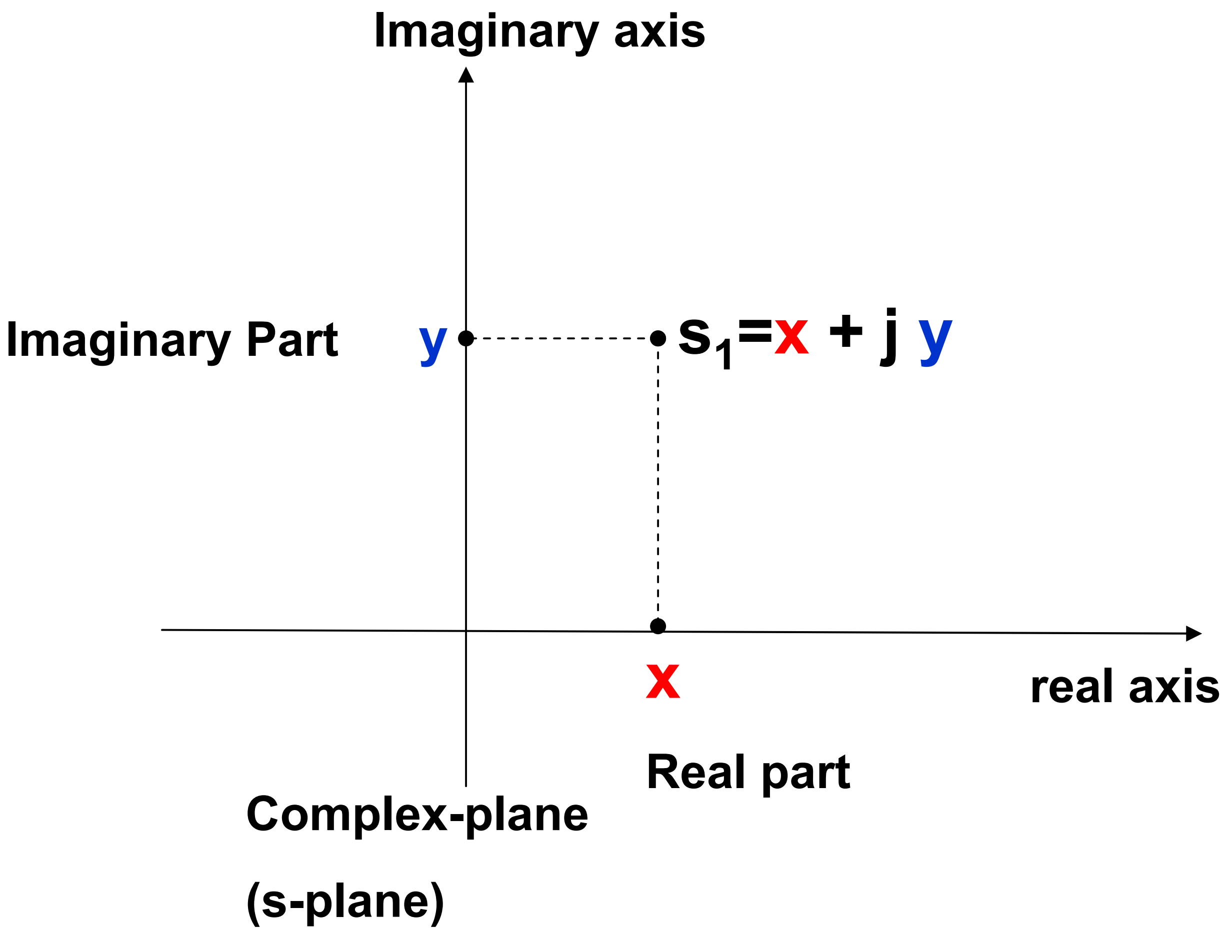
Two complex numbers z_1 and z_2 are equal if and only if their respective real and imaginary parts are equal

$z_1 = x_1 + j\ y_1$; $\qquad z_2 = x_2 + j\ y_2$

$$z_1 = z_2 \quad \Leftrightarrow \quad \begin{cases} x_1 = x_2 \\ y_1 = y_2 \end{cases}$$

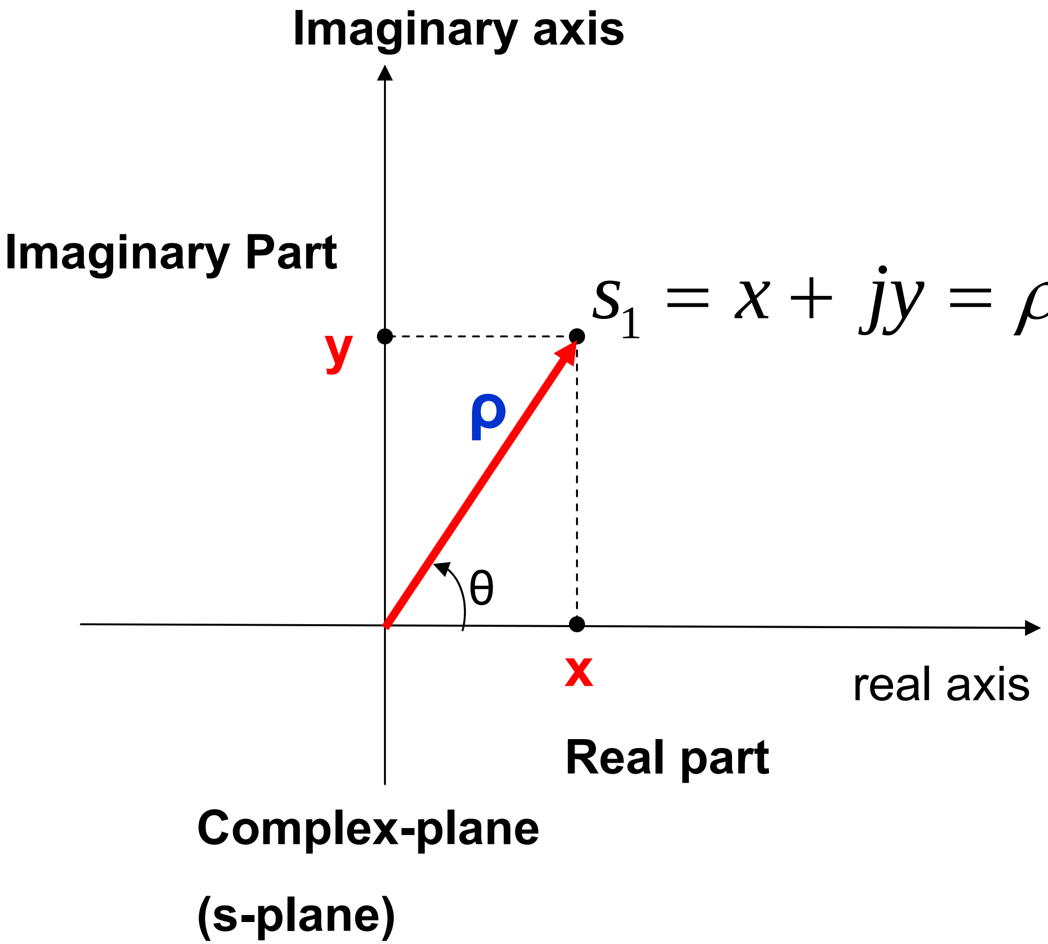
Representing Complex numbers

Rectangular representation



Representing Complex numbers

Polar representation



$$s_1 = x + jy = \rho e^{j\theta}$$

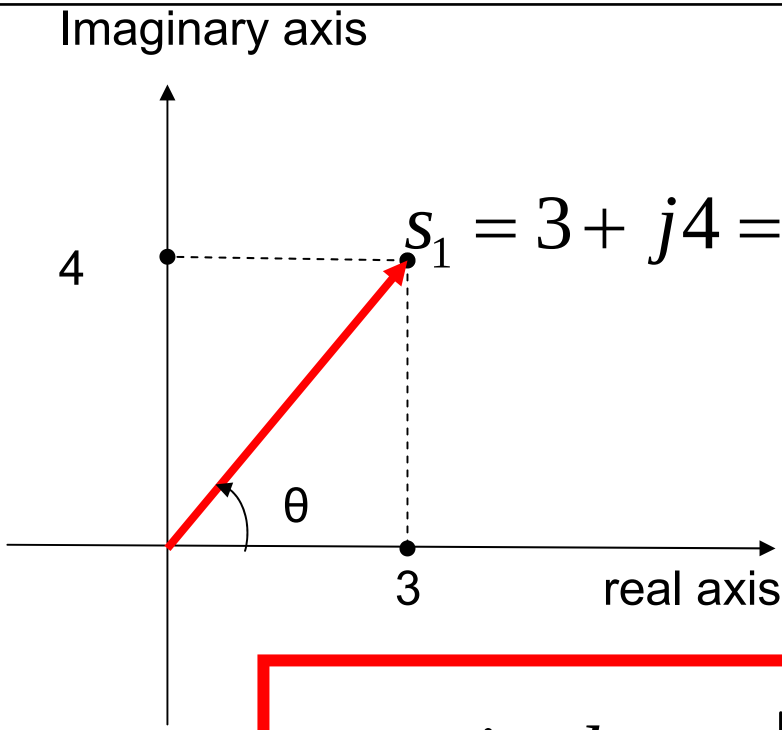
$\rho : length\, of\ s_1$

$\rho = |s_1|$

$\theta : phase\ \ angle$

Conversion between Representations

Example



$$s_1 = 3 + j4 = 5e^{j\,0.9273}$$

$$\theta = \tan^{-1}\left(\frac{\text{imaginary part}}{\text{real part}}\right)$$

$$magnitude \quad = |s_1| = \sqrt{3^2 + 4^2} = 5$$

$$phase \;\; angle = \theta = 0.9273 \; (radian)$$

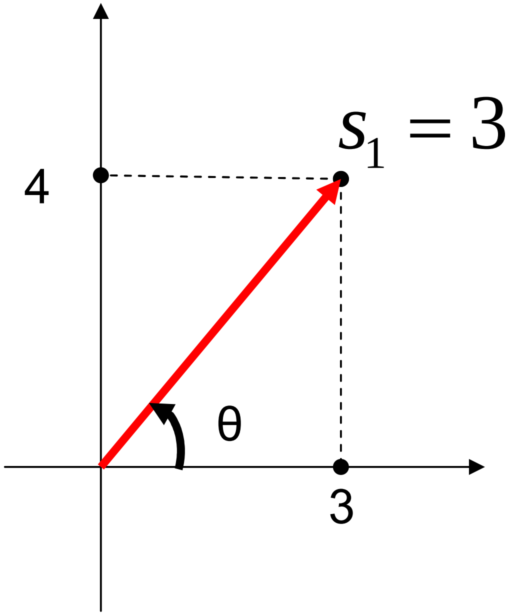
Euler Formula

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 0.9273$$

$$s_1 = 3 + j4 = 5e^{j\,0.9273}$$

$$= 5\cos(0.9273) + \text{j}\ \ 5\sin(0.9273)$$



Complex Numbers

Addition /Subtraction

$$z_1 = x_1 + j\ y_1 ; \qquad z_2 = x_2 + j\, y_2$$

$$z_1 + z_2 = (x_1 + x_2) + j\,(y_1 + y_2)$$

$$z_1 - z_2 = (x_1 - x_2) + j\,(y_1 - y_2)$$

$$(2 - j3) + (5 + j4) - 8 = (2 + 5 - 8) + j(-3 + 4 - 0)$$

$$= -1 + j$$

Complex Numbers

Multiplication/Division

$$z_1 = x_1 + j\ y_1 = r_1\, e^{j\theta_1}; \qquad z_2 = x_2 + j\, y_2 = r_2\, e^{j\theta_2}$$

$$z_1\ z_2 = \ (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1)$$

$$\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + j\,\frac{y_1 x_2 - y_2 x_1}{x_2^2 + y_2^2}$$

$$z_1\ z_2 = \ r_1 r_2\ e^{j(\theta_1 + \theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\ e^{j(\theta_1 - \theta_2)}$$

Operations

Examples

$$s = -1 + j3$$

$$z = 2 + j5$$

$$s + z = (-1 + 2) + j(3 + 5) = 1 + j8$$

$$s\,z = \left(-1 + j3\right)\left(2 + j5\right) = -17 + j$$

$$\frac{s}{z} = \frac{\left(-1 + j3\right)}{\left(2 + j5\right)} = \frac{13 + j11}{29}$$

More Examples

$$s = 2e^{j2}$$

$$z = 3e^{-j}$$

$$s\,z = 2e^{j2}\;\;3e^{-j} = (2 \bullet 3)e^{j(2-1)} = 6e^{j}$$

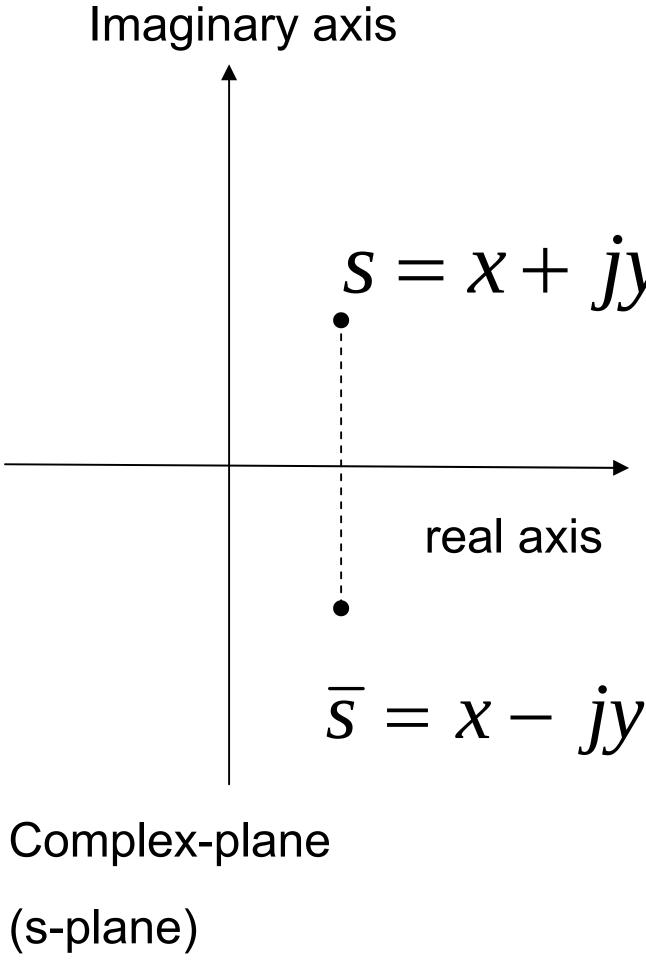
$$\frac{s}{z} = \frac{2e^{j2}}{3e^{-j}} = \frac{2}{3}e^{3j}$$

Conjugate

$$s = x + jy$$

$\bar{s}$: complex conjugate of s

$$\bar{s} = x - jy$$



Conjugate

$$s = x + jy$$

$$\bar{s} = x - jy$$

$$|s| = |\bar{s}| = \sqrt{x^2 + y^2}$$

$$|s|^2 = s\,\bar{s} = x^2 + y^2$$

Conjugate

$$s = -1 + j2$$

$$\bar{s} = -1 - j2$$

$$|s| = |\bar{s}| = \sqrt{x^2 + y^2} = \sqrt{5}$$

$$|s|^2 = s\ \bar{s} = x^2 + y^2 = 5$$

Conjugate real/imaginary part

$$s = x + jy$$

$$\bar{s} = x - jy$$

$$\mathrm{Re}\{s\} = x = \frac{(s + \bar{s})}{2}$$

$$\mathrm{Im}\{s\} = y = \frac{(s - \bar{s})}{2j}$$

Operations Polar coordinate Multiplication/Division

$$s_1 = \rho_1 e^{j\theta_1}, \qquad s_2 = \rho_2 e^{j\theta_2}$$

$$s_1\ s_2 = \left(\rho_1 e^{j\theta_1}\right)\left(\rho_2 e^{j\theta_2}\right) = \rho_1 \rho_2 e^{j\left(\theta_1 + \theta_2\right)}$$

$$\frac{s_1}{s_2} = \frac{\rho_1 e^{j\theta_1}}{\rho_2 e^{j\theta_2}} = \frac{\rho_1}{\rho_2} e^{j\left(\theta_1 - \theta_2\right)}$$

An example

$$(2 + j2) * (1 - j) = (4 + 0j)$$

$$2(1 + j) * (1 - j) = 2(1 + 1) = 4$$

$$2\sqrt{2}\ e^{j\pi/4}\ \sqrt{2}\ e^{-j\pi/4} = 4\ e^{j0} = 4$$

Keywords

- Conjugate
- Modulus
- Real part
- Imaginary part
- Polar coordinates
- Complex plane
- Imaginary axis
- Pure imaginary