H-infinity Error Bounds in Approximating Time-Delay Systems

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Abstract

In this paper, upper and lower bounds on approximating time-delay systems are proposed. Bounds on the infinity norm of the weighted error are obtained when the approximating function is a general rational function, all-pass function, Pade' and Laguerre approximations. In addition, approximations of the weighted errors for both Pade' and Lagurre approximating functions are developed. Examples are presented in illustration.

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1 Introduction

A transfer function of the form

$$G(s) = P(s)e^{-sd} \tag{1}$$

may be used to represent many dynamical systems. In (1), $P(s)$ represents the dynamics of the system, and e^{-sd} represents the input delay.

In many situations it is desirable to approximate the infinite-dimensional time-delay systems by a finite-dimensional rational transfer functions. Several techniques are available for approximating time-delay systems. Padé approximation has been widely used to approximate e^{-sd}(See, for example, [1]). Formulas for Padé approximation of any desirable order is available together with error bounds in the L_2 and L_∞ sense [1]. Hankel approximation of $P(s)e^{-sd}$ with stable and strictly proper $P(s)$ was obtained in [3]. In this case, better approximation was obtained but with a much larger computation burden. Methods based on truncation of Fourier-Laguerre series were developed in [4], and [6]. The methods are extended to a larger class of time-delay systems and are computationally efficient. However, the resulted finite-dimensional approximation may be of considerably high order and further model reduction is needed. Yoon and Lee [8], obtained rational approximation of e^{-sd} and $P(s)e^{-sd}$ based on truncated Blascke product together with L_2 and L_∞-norm bounds.

The error bounds are valuable in assessing the quality of the approximation. They can be used as a guide in the selection of the order of the approximating function. Several bounds are available. In this paper, bounds on the infinity-norm of the weighted approximation error are derived for different types of approximating functions: the general rational functions, the all-pass functions, the Padé and Laguerre approximations.

In the following section, the statement of the problem is presented. Some preliminary results are provided in Section 3. The main results are reported in section 4. An illustrative example is given in Section 5 and conclusion is given in Section 6.

2 Problem Statement

Let H_∞ denote the hardy space of functions analytic in the right hand side and bounded on the j-ω axis with the norm defined as

$$\|F\|_\infty = ess \sup_{\omega \in R} |F(j\omega)|$$

Our objective is to obtain bounds on the error introduced in approximating time-delay systems. The system under study is assumed to be a single-input single-output system described by (1) where $P(s)$ is assumed to be rational, stable and strictly proper. The approximate of $G(s)$ is given by $P(s)G_r(s)$ where $G_r(s)$ is an r th order rational approximation of e^{-sd} obtained such that

$$\epsilon = \left\| \left(G_r(sd) - e^{-sd} \right) W(s) \right\|_\infty \tag{2}$$

is reasonably small. The weighting function $W(s)$ is assumed to have the following form

$$W(s) = M \left(\frac{1}{1 + \tau s} \right)^k \tag{3}$$

where $k \geq 1$. The parameters k, M and τ are selected such that

$$|P(j\omega)| \leq \frac{M}{|1 + j\omega\tau|^k} \qquad \forall \omega$$

Renaming the variables and re-arranging, one can easily show that the error, ϵ, given by (2) is equivalent to

$$\epsilon = \left\| \left(G_r(s) - e^{-s} \right) W(\frac{s}{d}) \right\|_\infty \tag{4}$$

The approximating function G_r is assumed to be in one of the following sets:

1. General rth order transfer functions of the form

$$G_r(s) = \frac{\sum_{i=0}^{r} b_i\, s^i}{1 + \sum_{i=1}^{r} a_i\, s^i} \tag{5}$$

2. All-pass rth order transfer functions of the form

$$G_r(s) = \frac{1 + \sum_{i=1}^{r} a_i(-s)^i}{1 + \sum_{i=1}^{r} a_i s^i} \tag{6}$$

3. An rth order Padé Approximations with a transfer function

$$G_r(s) = \frac{1 + \sum_{i=1}^{r} a_i(-s)^i}{1 + \sum_{i=1}^{r} a_i s^i} \tag{7}$$

   where a_i are given by [1]

$$a_i = \frac{(2r-i)!r!}{(2r)!i!(r-i)!}$$

4. An rth order Laguerre approximation [6]

$$G_r(s) = \left[\frac{1 - \frac{s}{2r}}{1 + \frac{s}{2r}}\right]^r \tag{8}$$

Bounds on the error when the approximating function belongs to one of the above classes will be derived.

3 Preliminary Results

In this section several preliminary results are presented. The results presented here will prove useful in deriving the main results in the following section.

Let $G_r(s)$ be a transfer function whose frequency response is $G_r(j\omega) = |G_r(j\omega)|\, e^{j\Phi(\omega)}$, then

$$\left|e^{-j\omega} - G_r(j\omega)\right| = \sqrt{1 + |G_r(j\omega)|^2 - 2\,|G_r(j\omega)|\cos(\Phi(\omega)+\omega)} \tag{9}$$

$$= \sqrt{(|G_r(j\omega)| - \cos(\Phi(\omega)+\omega))^2 + \sin^2(\Phi(\omega)+\omega)} \tag{10}$$

and if $G_r(s)$ is an all-pass transfer function, then

$$\left|e^{-j\omega} - G_r(j\omega)\right| = \sqrt{(1-\cos(\Phi(\omega)+\omega))^2 + \sin^2(\Phi(\omega)+\omega)} \quad (11)$$

$$= \sqrt{2 - 2\cos(\Phi(\omega)+\omega)} \quad (12)$$

Proof: Using Euler identity for $e^{-j\omega}$ and manipulating the expression will result in (9). Equation (10) is obtained by adding and subtracting $\cos(\Phi(\omega)+\omega)$ and simplifying the resulted expression. The expression of the error for the all-pass case are obtained by direct substitution of $|G_r(j\omega)| = 1$ in (9) and (10).

The following results can be easily derived from Lemma 1.

Let $G_r(j\omega) = |G_r(j\omega)|\, e^{j\Phi(\omega)}$, if $\Phi(\omega_1) + \omega_1 \geq \frac{\pi}{2}$ then there exist $\omega_0 \leq \omega_1$ such that $|e^{-j\omega_0} - G_r(j\omega_0)| \geq 1$

Proof: If $\Phi(\omega_0)+\omega_0 = \frac{\pi}{2}$, then $|e^{-j\omega_0} - G_r(j\omega_0)| = \sqrt{|G_r(j\omega_0)|^2 + 1} \geq 1$.

If $G_r(s)$ is an all-pass transfer function with a phase angle $\Phi(\omega)$, and if $\Phi(\omega_1) + \omega_1 \geq \pi$, then there exist $\omega_0 \leq \omega_1$ such that $|e^{-j\omega_0} - G_r(j\omega_0)| = 2$.

Proof: When $\Phi(\omega_0) + \omega_0 = \pi$ then $|e^{-j\omega_0} - G_r(j\omega_0)| = \sqrt{2 - 2\cos(\pi)} =$ 2.

The following is an important result from linear control theory.

Fact 1: The phase angle $\Phi(\omega)$ of an rth order transfer function satisfies the following bound for all ω

$$\Phi(\omega) \geq \begin{cases} -r\pi & if G_r is non-minimum phase of the forms given in (5)-(8) \\ -\frac{r\pi}{2} & if G_r is minimum phase of the form given in (5) \end{cases}$$

4 Main Results

In this section, bounds on the weighted approximation error will be derived. We start by presenting the lower bounds.

4.1 Error Bounds.

Theorem 4.1 *Let e^{-sd} be approximated by an r^{th} order transfer function $G_r(s)$ then*

$$\left\| \left(e^{-sd} - G_r\right) \frac{M}{(1+\tau s)^k} \right\|_\infty \geq M\ \gamma \left[1 + \frac{1}{\beta}\left(r\pi + \frac{\pi}{\alpha}\right)^2 \left(\frac{\tau}{d}\right)^2\right]^{-\frac{k}{2}}$$

where

1. $\alpha = 2,\ \beta = 1,\ \gamma = 1$ $\qquad fornonminimumphaseapproximation$
2. $\alpha = 1,\ \beta = 1,\ \gamma = 2$ $\qquad forall - passapproximation$
3. $\alpha = 1,\ \beta = 4,\ \gamma = 1$ $\qquad forminimumphaseapproximation$

Proof: *For part 1, Fact 1 implies that $\Phi(\omega) \geq -r\pi$, and therefore $\Phi(\omega_1) + \omega_1 \geq \frac{\pi}{2}$ for all $\omega_1 \geq r\pi + \frac{\pi}{2}$. Using Corollary 1, there exist an $\omega_0 \leq r\pi + \frac{\pi}{2}$ such that*

$$\left|e^{-j\omega_0} - G_r(j\omega_0)\right| = 1.$$

Using the fact that $\frac{M}{\left(1+j\omega\frac{\tau}{d}\right)^k}$ is monotonically decreasing function of ω, we have

$$\left| \left(e^{-j\omega_1} - G_r(j\omega_0)\right) \frac{M}{\left(1 + j\omega_0\frac{\tau}{d}\right)^k} \right| \geq \frac{M}{\left[\sqrt{1 + \left(r\pi + \frac{\pi}{2}\right)^2 \left(\frac{\tau}{d}\right)^2}\right]^k}$$

To prove Part 2, Corollary 2 is used. There exist $\omega_0 \leq r\pi + \pi$ such that

$$\left|e^{-j\omega_0} - G_r(j\omega_0)\right| = 2,$$

and the rest follows the proof of part 1. For the third part, Fact 1 and Corollary 1 are used. There exist $\omega_0 \leq \frac{r\pi}{2} + \frac{\pi}{2}$ such that $e^{-j\omega_0} - G_r(j\omega_0) = 1$, the remaining of the proof follows that of part 1.

An upper bound may be obtained by replacing the original weight in (3) by

$$M\left|\frac{1}{\tau s}\right|^k$$

Note that

$$M\left|\frac{1}{1+j\omega\tau}\right|^k \leq M\left|\frac{1}{j\omega\tau}\right|^k$$

We now present the following corollary.

For all pass systems,

$$\left\|\left(e^{-sd}-G_r(sd)\right)\frac{M}{(1+\tau s)^k}\right\|_\infty \leq \max_\omega \frac{\sqrt{2-2\cos(\omega+\Phi(\omega))}M}{(\omega)^k}\left(\frac{d}{\tau}\right)^k$$

Proof:

$$\begin{aligned}\left|\left(e^{-j\omega}-G_r(j\omega)\right)\frac{M}{\left(1+j\frac{\tau}{d}\omega\right)^k}\right| &= \sqrt{2-2\cos(\omega+\Phi(\omega))}M\left|\frac{1}{1+j\omega\frac{\tau}{d}}\right|^k \\ &\leq \sqrt{2-2\cos(\omega+\Phi(\omega))}M\left|\frac{1}{j\omega\frac{\tau}{d}}\right|^k\end{aligned}$$

Now, taking the maximum of the right hand side gives the above result.

4.2 Error Bounds for Padé and Laguerre Approximations

The structure of the Padé and Laguerre approximations are well known and therefore we expect that tighter bounds may be obtained. Consider an rth order all-pass transfer function $G_r(s)$ with a phase angle $\Phi(\omega)$. The following inequality is valid for all θ,

$$\left\|\left(e^{-sd}-G_r(sd)\right)\frac{M}{(1+\tau s)^k}\right\|_\infty \geq M\sqrt{\frac{2-2\cos(\theta)}{\left(1+(\omega_r^*)^2\left(\frac{\tau}{d}\right)^2\right)^k}} \tag{13}$$

where ω_r^* is the smallest frequency that satisfies

$$\omega_r^*+\Phi(\omega_r^*)=\theta$$

Note that ω_r^* depends on the family of the approximating function and on the selected order. Determining the largest lower bound may not be easy. However in the following subsection we give an approximate behavior of the right hand side of (13).

Theorem 4.2 *If $G_r(sd)$ is an r th order Laguerre approximation of e^{-sd}, then for $k \leq 3$, the following is true*

$$\left\| \left(e^{-j\omega d} - G_r(j\omega d)\right) \frac{M}{(1+j\omega\tau)^k} \right\|_\infty \leq \frac{2M}{\sqrt{1+\left(\underline{\omega}\frac{\tau}{d}\right)^2}}$$

where

$$\underline{\omega} = \min\left\{2r, \left(6\pi r^2\right)^{1/3}\right\}$$

: The phase angle of $G_r(s)$ is

$$\Phi(\omega) = -2r \arctan(\frac{\omega}{2r})$$

If $\frac{\omega}{2r} < 1$, The Taylor series expansion of $\Phi(\omega) + \omega$ is given by

$$\Phi(\omega) + \omega = 2r \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)} \left(\frac{\omega}{2r}\right)^{2k+1}$$

The alternating series convergences for $\frac{\omega}{2r} < 1$ and

$$|\Phi(\omega) + \omega| \leq \frac{\omega^3}{12r^2}$$

Using the standard inequality

$$\frac{2\sqrt{2}}{\pi} |\theta| \leq \left|1 - e^{j\theta}\right| \leq |\theta| \qquad if \;\; |\theta| \leq \frac{\pi}{2}$$

gives us the following bound

$$\left|e^{-j\omega} - G_r(j\omega)\right| \leq \frac{\omega^3}{12r^2} \qquad if \quad \omega \leq \min\left\{2r, \left(6\pi r^2\right)^{1/3}\right\} = \underline{\omega}$$

$$\left|e^{-j\omega} - G_r(j\omega)\right| \leq \begin{cases} \frac{\omega^3}{12r^2} & if \quad \omega \leq \underline{\omega} \\ 2 & otherwise \end{cases}$$

so,

$$\left| \left(e^{-j\omega} - G_r(j\omega)\right) \frac{M}{\left(1+j\omega\frac{\tau}{d}\right)^k} \right| \leq \begin{cases} \frac{\omega^3}{12r^2} \frac{M}{\left(1+\left(\omega\frac{\tau}{d}\right)^2\right)^{k/2}} & if \quad \omega \leq \underline{\omega} \\ \frac{2M}{\left(1+\left(\omega\frac{\tau}{d}\right)^2\right)^{k/2}} & otherwise \end{cases}$$

Examining the derivative of the error, $\frac{\omega^3}{12r^2}\frac{M}{\left(1+\left(\omega\frac{\tau}{d}\right)^2\right)^{k/2}}$, *we conclude that the derivative is positive for all* $\omega \leq \underline{\omega}$ *and* $k \leq 3$. *This means that the error is monotonically increasing in the interval* $[0, \underline{\omega}]$, *which makes us conclude that the peak of the weighted error occurs at* $\omega \geq \underline{\omega}$. *Since the weight is monotonically decreasing, the error will not exceed the product of the maximums. Hence the following is true for all* ω.

$$\left|\left(e^{-j\omega} - G_r(j\omega)\right)\frac{M}{\left(1+j\omega\frac{\tau}{d}\right)^k}\right| = \left|\left(e^{-j\omega} - G_r(j\omega)\right)\right|\left|\frac{M}{\left(1+j\omega\frac{\tau}{d}\right)^k}\right| \leq \frac{2M}{\left(1+\left(\underline{\omega}\frac{\tau}{d}\right)^2\right)^{k/2}}$$

Now the result of the theorem follows.

4.3 Approximation of the weighted Error

Since the formulas of the Padé and Laguerre approximations are known, one can easily obtain tighter upper and lower bounds using curve fitting algorithms. In this section, curve fitting will be used to find approximate behavior of (13), and approximations of the infinity-norm of the weighted approximation error for both Padé and Laguerre.

From experimenting with many examples, it is observed that the largest bound in (13) is obtained when $\theta \approx \pi$. Fixing $\theta = \pi$, the corresponding ω_r^* can be calculated. The values of ω_r^* for different orders are given in Table 1.

Table1:Thesmallestfrequenciesatwhichtheunweightederroris2,$M = k = 1$

r	1	2	3	4	5	6	7	8	9	10
ω_r^*	5.595	7.917	10.175	12.393	14.585	16.757	18.193	21.057	23.191	25.317

The frequency ω_r^* can be accurately represented by a second-order polynomial

$$\omega_r^* = -0.0047r^2 + 2.2297r + 3.4928$$

Now, this approximating behavior together with the above lower bound on the approximation error will produce the following approximation on the behavior of the lower bound. It is a function of the time-delay and the order of the approximating transfer function.

$$\left\| \left(e^{-sd} - G_r\right) \frac{M}{(1+\tau s)^k} \right\|_\infty \geq \frac{2}{\sqrt{\left(1 + (-0.0047r^2 + 2.2297r + 3.4928)^2 \left(\frac{\tau}{d}\right)^2\right)^k}} \tag{14}$$

Similarly, a lower bound on Laguerre approximation can be obtained by finding and expression for ω_r^* from the data in table 2.

Table 2: The smallest frequencies at which the unweighted error is 2.

r	1	2	3	4	5	6	7	8	9	10
ω_r^*	5.597	7.455	9.056	10.499	11.834	13.086	14.272	15.405	16.493	17.542

A good approximation of ω_r^* is given by

$$\omega_r^* = (6.6717r + 7.0805)^{2/3}$$

which gives rise to the following bound

$$\left\| \left(e^{-sd} - G_r\right) \frac{M}{(1+\tau s)^k} \right\|_\infty \geq \frac{2}{\sqrt{\left(1 + (6.6717r + 7.0805)^{4/3} \left(\frac{\tau}{d}\right)^2\right)^k}}$$

The $\infty-$ norm of the weighted error for $W(s) = \frac{1}{(1+\tau s)^2}$ and $G_r(sd)$ can be generated for a range of values of τ, d and r. However a plot of the exact error for the range $r \in [1, 20]$ and $\frac{\tau}{d} \in [0, 5]$ suggests that the behavior can be described as follows. For the rth order Pade approximation, the weighted infinity-norm error may be fitted as

$$\left\| \frac{\left(G_r(sd) - e^{-sd}\right)}{(1+\tau s)^2} \right\|_\infty \approx \frac{1}{\left|(0.0062r^2 - 0.1010r + 0.4848)\frac{\tau}{d} + (-0.0197r^2 + 1.8359r + 1.0637)\right|^2} \tag{15}$$

and

$$\left\| \frac{\left(G_r(sd) - e^{-sd}\right)}{(1+\tau s)^2} \right\|_\infty \approx \frac{1}{\left|(0.0041r^2 - 0.0729r + 0.4764)\frac{\tau}{d} + (-0.0351r^2 + 1.2894r + 1.6376)\right|^2} \quad (16)$$

for the rth order Laguerre approximation. The deviations from the true infinity-norm of the above formulas is less than 0.01 over the range $\frac{\tau}{d} \leq 5$ and $r < 20$.

5 Illustrative Examples

5.1 Example 1:

It is required to use Padé and Laguerre approximations to obtain rational approximants of $\frac{e^{-s}}{(1+s)^2}$. The rth order approximation $G_r(s)$ is obtained so that the error

$$\left\| \left(e^{-s} - G_r(s)\right) \frac{1}{(1+s)^2} \right\|_\infty$$

is reasonably small. The actual error, the predicted error provided by (15), and (16), lower bounds of theorem 1 and the one given by (14) together with the upper bound for Laguerre are all listed in Table 3 for orders 1 to 10.

Table3:Theatucalerroranddifferenterrorbounds

	Pade				Laguerre		
Order r	Actual Error	Lower Bound 1	Lower Bound 2	predicted Error	Actual Error	Predicted Error	Upper Bound
1	0.0989	0.0862	0.0594	0.0935	0.0989	0.0919	0.4000
2	0.0403	0.0319	0.0313	0.0406	0.0502	0.0511	0.1176
3	0.0225	0.0164	0.0193	0.0227	0.0325	0.0332	0.0632
4	0.0146	0.0100	0.0131	0.0146	0.0235	0.0238	0.0435
5	0.0103	0.0067	0.0094	0.0102	0.0182	0.0182	0.0325
6	0.0076	0.0048	0.0071	0.0076	0.0147	0.0145	0.0256
7	0.0059	0.0036	0.0056	0.0059	0.0122	0.0120	0.0209
8	0.0047	0.0028	0.0045	0.0047	0.0104	0.0103	0.0175
9	0.0039	0.0022	0.0037	0.0039	0.0090	0.0090	0.0150
10	0.0032	0.0018	0.0031	0.0033	0.0079	0.0080	0.0130

Several observations can be made in examining table 1. It is obvious that the actual error for the Lagurre is larger comapred to the that of the Pade approximation. This makes the upper bound of the Lagurre approximations usable as an upper bound for the Pade approximation. When we comapre the upper bounds developed in this paper for the Lagurre approximation aginest the bounds for Pade in [2], we see that the developed bounds are tighter. Another observation is that the predicted error for both the Pade and the Lagurre approximations are very accurate.

5.2 Example 2:

This example was used in [2] and an upper bound for the error was given by

$$\left\| (e^{-ds} - G(s)) \frac{1}{s\tau + 1} \right\|_\infty \leq \frac{d/\tau}{r\left(\sqrt{2}/e\right)^{\frac{1}{2}}}$$

Similar to the situation in Example 1, it was observed that the actual error for the Lagurre is larger than that of Pade. This makes it possible to use the upper bound of the Lagurre as an upper bound for the Pade. The upper bound given in Theorem 2 for the Lagurre approximation is given by

$$\left\|(e^{-ds} - G(s))\frac{1}{s\tau+1}\right\|_\infty \leq \frac{1}{\sqrt{1+\left(\underline{\omega}\frac{\tau}{d}\right)^2}} where \underline{\omega} = \min\left\{2r, \left(6\pi r^2\right)^{\frac{1}{3}}\right\}$$

We can show that this upper bound is smaller than the one given in [2] for all values of $r \leq 50$. This makes the upper bound developed in this paper tighter that the one in [2]. However, it will be interesting to develop an upper bound on the Pade approximation along the same lines developed in Theorem 2. This needs a neat expression for the phase of Pade approximations.

6 Conclusions

In this paper, lower and upper bounds on approximation of time-delay systems are obtained. Bounds on the error when the approximating function is an r^{th} order minimal phase, non-minimal phase, all pass, Padé and Laguerre transfer functions are given. Fitting formulas for the weighted infinity-norm error of Padé and Laguerre approximations are also obtained.

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References

[1] Lam, J., "*Model Reduction of Delay Systems Using Padé's Approximation*", Int. J. Control, vol. 57, pp 377-391, 1987.

[2] Glader, C., Hognas, G., Mäkilä, P.M. and Toivonen H.T., "*Approximation of Delay systems-A case Study*", Int. J. Control 33, pp. 369-390, 1991.

[3] Glover, K., Lam, J. and Partington, J. R., "*Rational approximation of a class of Infinite-dimensional system I: Singular value of Hankel Operator*". Math. Control Circ. Syst., Vol. 3, pp. 325-344, 1990.

[4] Gu, G., Khargonekar, P. and . Lee, E.B, "*Approximation of Infinite Dimensional Systems*", IEEE Trans. Auto. Control, pp. 610-618, 1989.

[5] Lam, J., "*Convergence of a Class of Padé's Approximation for Delay Systems*", Int. J. Control, vol. 52, pp 989-1008, 1990.

[6] Mäkilä, P. M., "*Laguerre Series Approximation of Infinite Dimensional Systems*", Automatica, vol. 26, pp 985-995, 1990.

[7] Partington, J. R., "*Approximation of Delay systems by Fourier-Laguerre Series*", Automatica vol 27, pp. 569-572, 1991.

[8] Yoon, M. G. and Lee B. H. "*A New Approximation Method for Time-Delay Systems*". IEEE Trans. Auto. Control, pp. 1008-1012, 1997.

[9] Mäkilä, P.M. and Partington, J. R., "Shifted Operator Induced Approximations of Delay Systems", SIAM J. Control Optim. Vol 37, pp. 1897-1912, 1990.