Review of Complex Variables

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Term 002

1 Introduction

The set of complex numbers are numbers that are expressed in the form

$$z = x + jy$$

where x and y are real numbers, x is called the real part[1] of z ($x = Re\{z\}$), y is the imaginary part of z ($y = Im\{z\}$) and ,$j = \sqrt{-1}$. Using complex numbers allows us to solve the following equation

$$x^2 + 1 = 0$$

which does not have a solution if we restrict x to be a real number.

1.1 Complex Arithmetic:

$$\begin{aligned} j^2 &= -1 \\ j^3 &= -j \\ j^4 &= 1 \end{aligned}$$

If a is a positive real number then

$$\begin{aligned} \sqrt{-a} &= j\sqrt{a} \\ \sqrt{-3}\sqrt{27} &= \sqrt{-81} = j9 \end{aligned}$$

$$\begin{aligned} (x + jy) + (u + jv) &= (x + u) + j(y + v) \\ (x + jy) - (u + jv) &= (x - u) + j(y - v) \\ (x + jy) \times (u + jv) &= (xu - yv) + j(xv + yu) \\ \frac{(x + jy)}{(u + jv)} &= \frac{(xu + yu) + j(yu - xv)}{u^2 + v^2} \end{aligned}$$

[1] If x=0, z is a real number
if y=0 then z is a pure imaginary number

1.2 Conjugate and Modulus

Definition 1 *The* ***conjugate***[2] *of a complex number* $z = x + jy$ *is defined as* $\bar{z} = x - jy$.

Note that

$$\begin{aligned}
Re(z) &= Re(\bar{z}) \\
Im(z) &= -Im(\bar{z}) \\
Re(z) &= \frac{1}{2}(z + \bar{z}) \\
Im(z) &= \frac{1}{2}(z - \bar{z}) \\
\overline{\bar{z} + \bar{w}} &= z + w \\
\bar{z}\ \bar{w} &= \overline{zw} \\
\overline{\bar{z}\ +\ w} &= z + \bar{w}
\end{aligned}$$

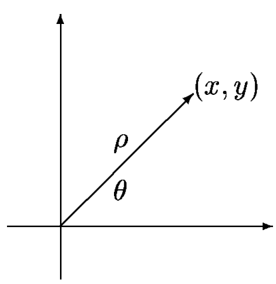
Definition 2 *The modulus*[3] *of a complex number* z *is defined as .*

$$|z| = |x + jy| = \sqrt{x^2 + y^2}$$

$$\begin{aligned}
|z| &= \sqrt{\bar{z}z} \\
|z| &= |\bar{z}| \\
\left|\frac{z}{w}\right| &= \frac{|z|}{|w|} \\
|zw| &= |z|\,|w| \\
|z + w| &\leq |z| + |w| \quad \textit{triangular inequality}
\end{aligned}$$

1.3 Polar Form

A complex number can be expressed in different forms: (standard, polar and trigonometric)



[2]some times the conjugate is denoted by z^*

[3]also called absolute value or length.

$$
\begin{aligned}
z &= x + jy = \rho e^{j\theta} = \rho\cos(\theta) + j\rho\sin(\theta) \\
\rho &= |z| = \sqrt{x^2 + y^2} \\
\theta &= \arg(z) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{Im(z)}{Re(z)}\right) \\
x &= \rho\cos(\theta) \\
y &= \rho\sin(\theta)
\end{aligned}
$$

ρ is the modulus of z and θ is called the argument of z.

Operations in Polar Form:

$$
\begin{aligned}
\rho_1 e^{j\theta_1}\ \rho_2 e^{j\theta_2} &= \rho_1\rho_2 e^{j(\theta_1+\theta_2)} \\
\frac{\rho_1 e^{j\theta_1}}{\rho_2 e^{j\theta_2}} &= \frac{\rho_1}{\rho_2} e^{j(\theta_1-\theta_2)}
\end{aligned}
$$

1.3.1 Some useful relationships:

$$
\begin{aligned}
j &= 1\ \underline{\lfloor \pi/2} \\
-j &= 1\ \underline{\lfloor -\pi/2} \\
z &= re^{j\theta} \quad \Rightarrow z^n = r^n e^{jn\theta} = r^n \quad \underline{\lfloor n\theta}
\end{aligned}
$$

Euler Formula

$$\rho e^{j\theta} = \rho\cos(\theta) + j\rho\sin(\theta)$$

DeMoivre Formula

$$(\cos(\theta) + j\sin(\theta))^n = \cos(n\theta) + j\sin(n\theta)$$

$$z = re^{j\theta} \quad \Rightarrow z^n = r^n e^{jn\theta} = r^n \quad \underline{\lfloor n\theta}$$

Example 1 *Express* $4 + j3$ *in polar form*

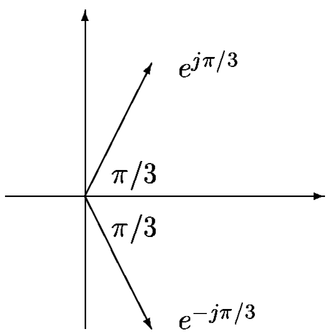
$$
\begin{aligned}
\rho &= \sqrt{4^2 + 3^2} = 5 \\
\theta &= \tan^{-1}\left(\frac{3}{4}\right) = 36.9^\circ
\end{aligned}
$$

Example 2 *If* $z = 4 + j5$ *what is* $\overline{z}$

$$\overline{4 + j5} = 4 - j5$$

If $z = 3e^{j\pi/3}$ what is $\overline{z}$

$$\overline{3e^{j\pi/3}} = 3e^{-j\pi/3}$$



1.4 The Poles and Zeros

.A complex function $F(s)$ is **analytic** in a region if the function and all its derivatives exists in that region. The points at which the function is analytic are called ordinary while points in the s-plane at which the function $F(s)$ are not analytic are called the singular points of $F(s)$. Singular points at which the function $F(s)$ approaches infinity are called the **poles** of $F(s)$. If $F(s)$ is a rational function then the **poles** are the zero of the denominator polynomial. Points at which $F(s)$ equals zero are called the zeros of $F(s)$. Finite **zeros** of a rational function $F(s)$ are the zeros of the numerator polynomial.

Example 3 *What are the poles and zeros of the following function*

$$F(s) = \frac{s+2}{(s+3)(s+4)}, \qquad G(s) = \frac{4}{(s+2)^2(s+5)}$$

The functions $F(s)$ has two simple poles (at $s=-3$ and $s=-4$) and one finite zero ($s=-2$). It has another zero at infinity.

$G(s)$ has double poles (repeated poles) at $s=-2$ and a simple pole at $s=-5$ and has no finite zero.

Definition 3 *A rational complex function $F(s) = \frac{N(s)}{D(s)}$ is called* **proper** *if*

$$degree(D(s)) \geq degree(N(s))$$

It is **strictly proper** *if*

$$degree(D(s)) > degree(N(s))$$