Term 081

CISE302: Linear Control Systems

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Chapter 7: Root locus

Learning Objectives

- Understand the concept of root locus and its role in control system design
- Be able to sketch root locus and use MATLAB to plot the root locus
- Recognize the role of root locus in parameter design and sensitivity analysis
- Be able to design controllers using root locus

Outlines

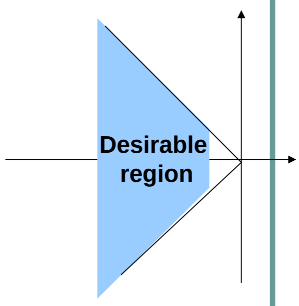
- The concept of root locus

Root locus

- The location of the roots of the characteristic equation determines the systems response.
- Modifying one or more of the system's parameter cause the roots of the characteristic equation to change.
- Root locus is a graphical method that determines the location of the roots as one parameter change.

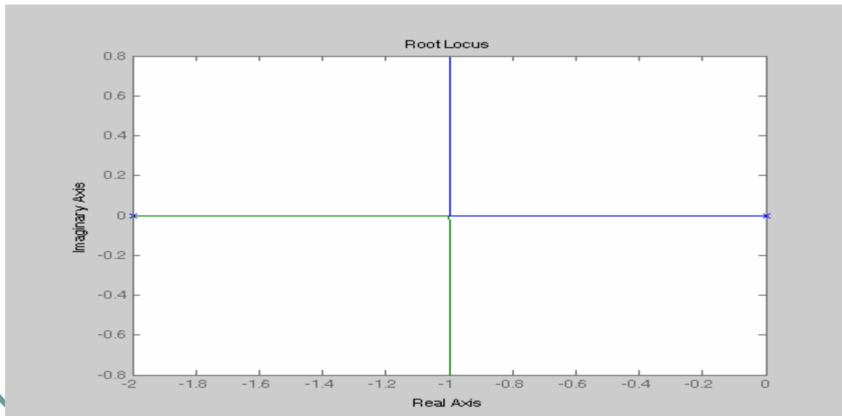
Root locus

- Root locus was developed by Evans in 1948.
- Used to analyze and design systems
- We use it to ensure that the roots of the characteristic equation are in a desirable region
- Can be used to study sensitivity of the pole location w.r.t changes in the parameter.



Root Locus Using MATLAB

- rlocus(1,[1 2 0])



Example 1

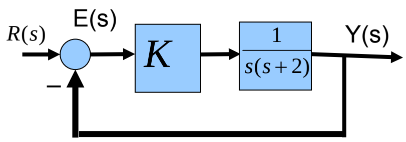
the characteristic equation

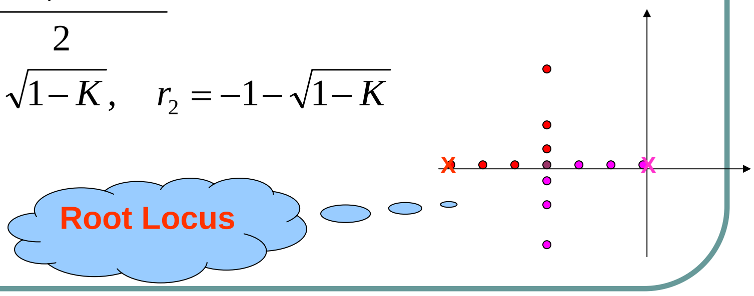
$$s^2 + 2s + K = 0$$

roots are:

$$r_{1,2} = \frac{-2 \pm \sqrt{4-4K}}{2}$$

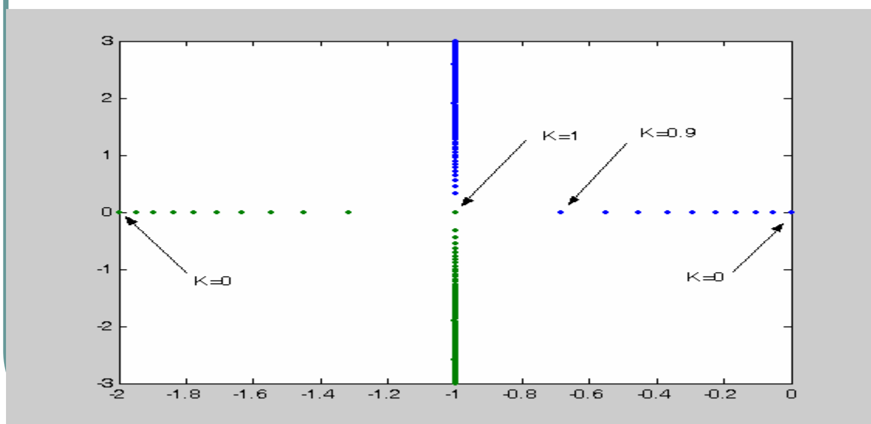
$$r_1 = -1 + \sqrt{1-K}, \quad r_2 = -1 - \sqrt{1-K}$$





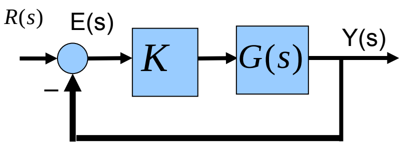
Root locus can be used to study sensitivity of system w.r.t variations in the parameter

- K=0:0.1:10; G=rlocus(1,[1 2 0],K); plot(G,'.')



Mathematics of root locus

C



$$1 + KG(s) = 0$$

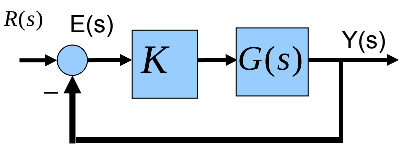
$$|KG| \angle KG(s) = -1 + j0$$

$$\Rightarrow |KG| = 1; \ \angle KG(s) = 180 \pm k360$$

k : integer

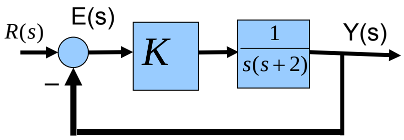
Question

C



How does the roots of the characteristic equation $1 + KG(s) = 0$ change as K varies between 0 and ∞?

Example



$$|KG(s)| = \left|\frac{K}{s(s+2)}\right| = 1$$

$$When\ K = 0 \quad \rightarrow s(s+2) = 0 \quad \rightarrow s = 0\ or\ s = -2$$

The locus of the roots of 1+KG(s)=0 starts at the poles of G(s)

Root Locus Procedure

- **Initial Step:**

Express the characteristic equation in the form

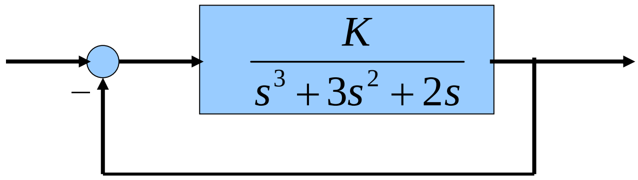
1+K P(s)= 0

Factor P(s) and rewrite the equation in the form

$$1 + K\frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)} = 0$$

Example 1

- Apply the initial step of the root locus procedure for the system

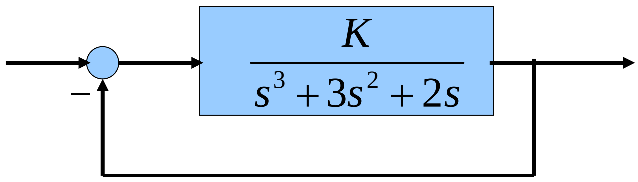


Example

- Step 1: Express the characteristics as

$$1 + K\ P(s) = 0$$

$$1 + K\frac{1}{s^3 + 3s^2 + 2s} = 0$$



Example

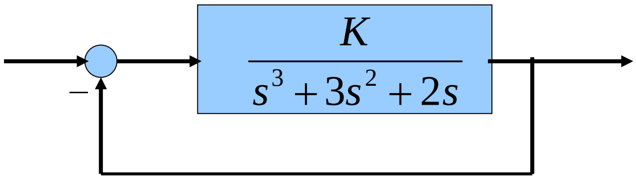
- Step 2: Factor P(s)

$$1 + K\ P(s) = 0$$

$$1 + K \frac{1}{s(s+1)(s+2)} = 0$$

n = 3

m = 0

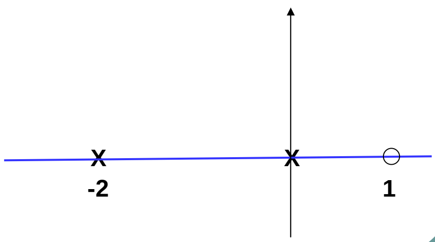


Root Locus Procedure

Locate the open loop poles and zeros of P(s)

- Locate the poles and zeros of P(s) on the s-plane.
- Use X for poles and O for zeros

$$1 + K \frac{(s-1)}{s(s+2)} = 0$$



- The root locus starts at the poles of P(s) and terminates at the zeros of P(s) or at infinity

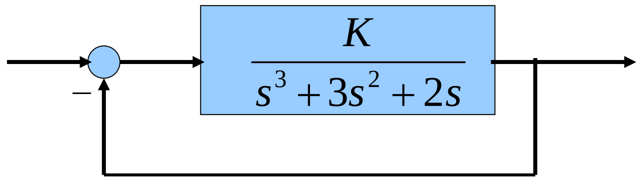
Root Locus Procedure
of separate root loci

- The number of separate root loci = The number of poles of P(s)

  Note that (# poles ≥ # zeros)

Example 2

- How many separate root loci does the root locus of the system has?



$$1 + K\ P(s) = 0$$

$$1 + K \frac{1}{s(s+1)(s+2)} = 0$$

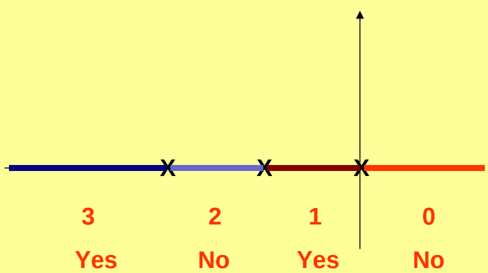
# of poles =3 → **Three separate root loci**

Root Locus Procedure

Segments of real line that is part of root locus

- **2: Segments of real line:**

A segment of real line is a part of root locus if sum of # of poles plus # of zeros to right of that segment is odd.



Example

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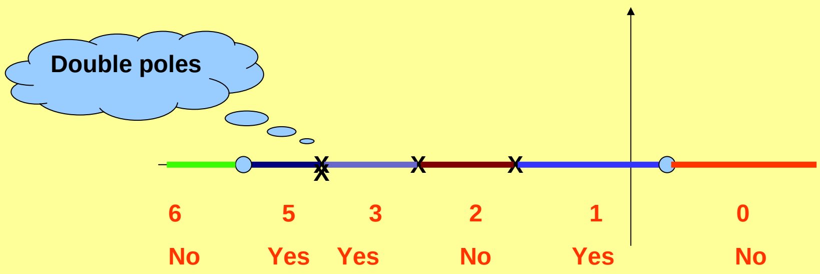
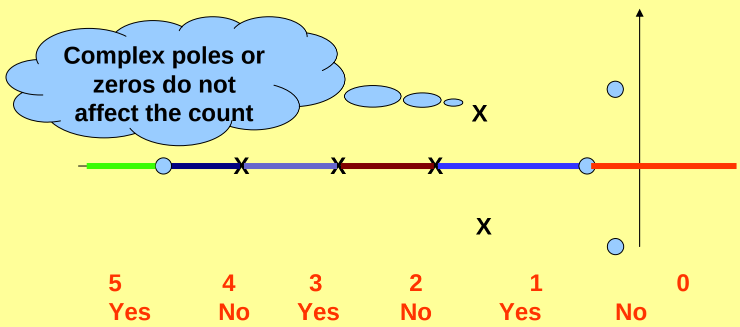


Example

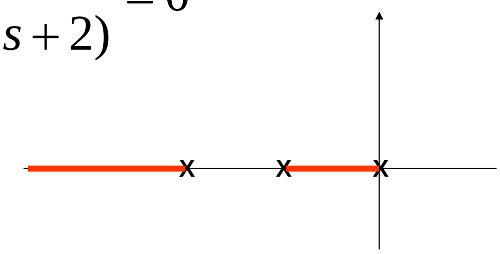
A segment of real line is a part of root locus if sum of # of poles plus # of zeros to right of that segment is odd.



Example

- Step 2: Segments of real axis that are parts of the root locus

$$1 + K\frac{1}{s(s+1)(s+2)} = 0$$



Asymptotes

- If n > m

# of poles of P(s) ># of zeros

The root locus approaches infinity at n-m directions.

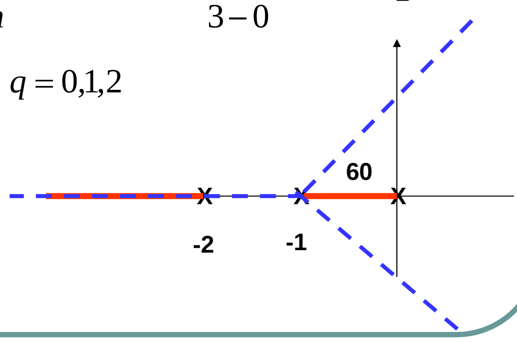
Asymptotes

- Step 3: # of asymptotes = n – m =3

$$centroid = \frac{\sum poles - \sum zeros}{n-m} = \frac{\{0-1-2\}-\{0\}}{3-0} = -1$$

$$\Phi_A = \frac{2q+1}{n-m}180 \qquad q = 0,1,2$$

$$\Phi_A = 60,\ 180,\ 300$$



Example

- How many asymptotes?

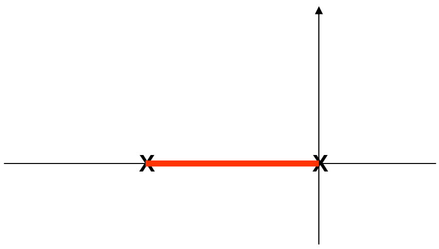
$$1 + K\frac{1}{s(s+1)} = 0$$

n=2, m=0, n-m=2

2 asymptotes

Centroid =(0+(-1))/2

=-0.5



Intersection with Imaginary axis

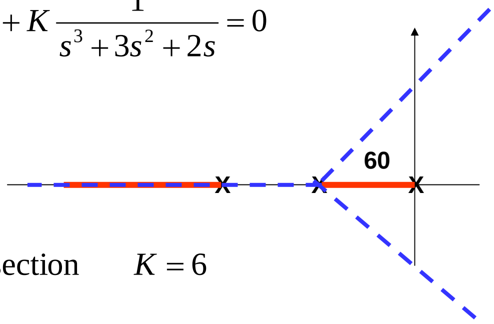
- To find intersection with imaginary axis use Routh Hurwitz method

Characteristic equation $1+K\dfrac{1}{s^3+3s^2+2s}=0$

$s^3+3s^2+2s+K=0$

s^3	1	2
s^2	3	K
s^1	$\dfrac{6-K}{3}$	
s^0	K	

Intersection $K=6$

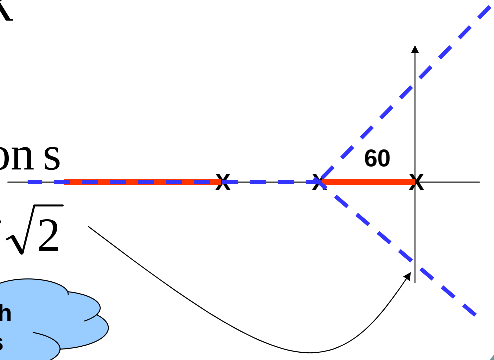


Intersection with Imaginary axis

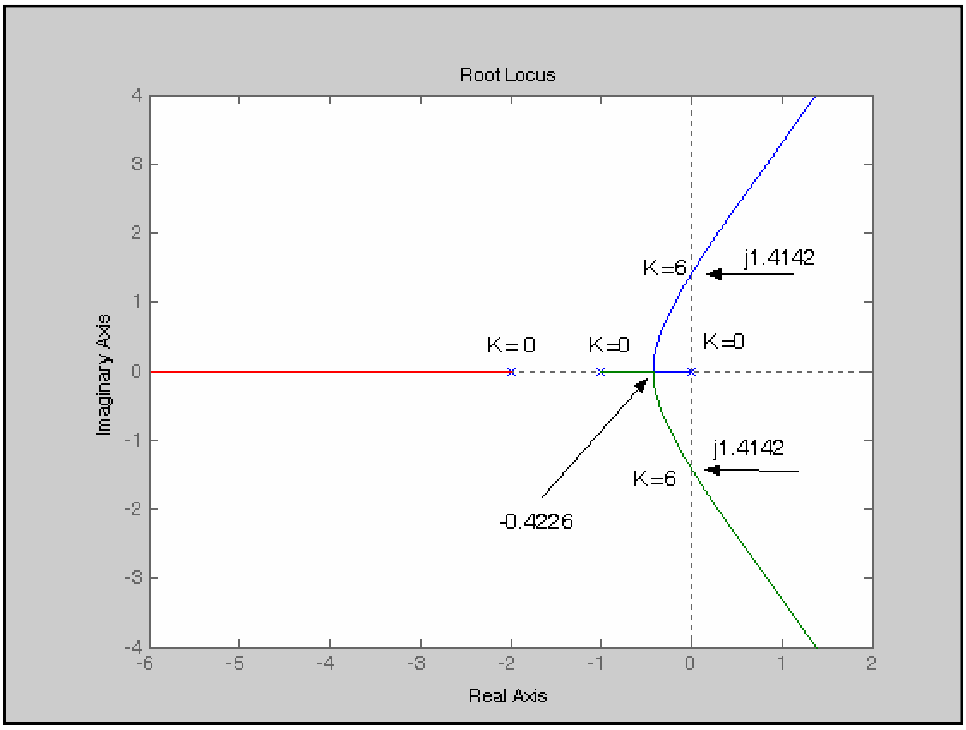
s^3	1	2
s^2	3	$6=K$
s^1	0	0

Auxiliary Equations

$3s^2 + 6 \Rightarrow s=\pm j\sqrt{2}$



Intersection with imaginary axis



Breakaway points

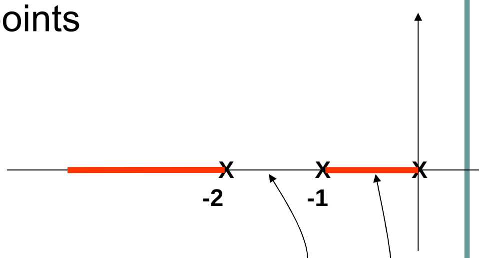
- Breakaway points are points at which the root loci breakaway from real axis or the root loci return to real axis.
- At breakaway points

$$\frac{dP(s)}{ds} = 0$$

- Solve the above equation to determine the breakaway point. Select solution that are in segments of real axis that is part of root locus

Breakaway points

- To find breakaway points



$$\frac{d}{ds}\left(\frac{1}{s^3+3s^2+2s}\right)=0$$

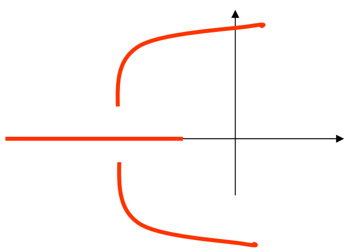
$$\frac{0-1\left(3s^2+6s+2\right)}{\left(s^3+3s^2+2s\right)^2}=0$$

break away points -1.5774 *and* -0.4226

Select -0.4226

Symmetry of root locus

- The root locus is symmetric with respect to the real axis.



Angle of departure

- Sum of angle contributions of poles and zeros (measured with standard reference) = 180+360k
- In the example all poles/zeros are real

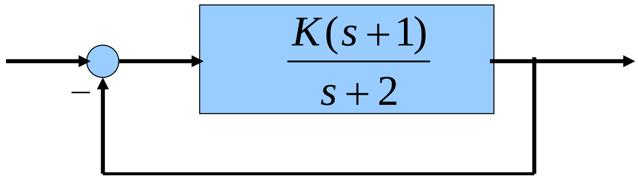
  No need to do this

Angle criteria

- At all points of the root loci : Sum of angle contributions of poles and zeros (measured with standard reference) = 180+360k

Example 8

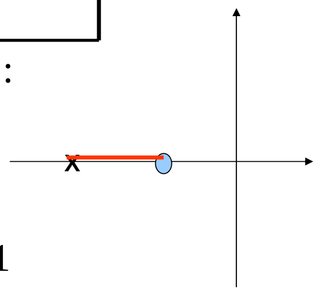
- Draw root locus for the following system

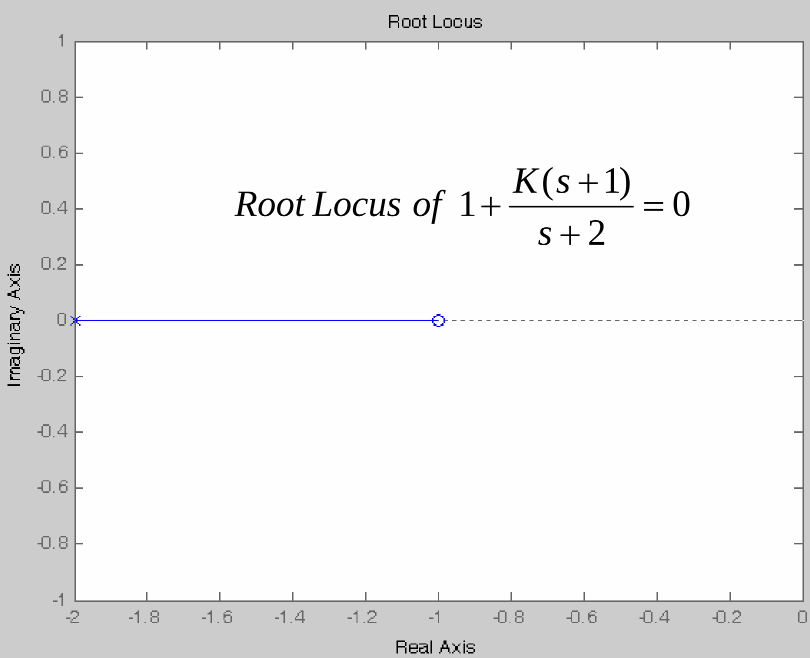


Characteristic Equation :

$$1+K\frac{s+1}{s+2}=0$$

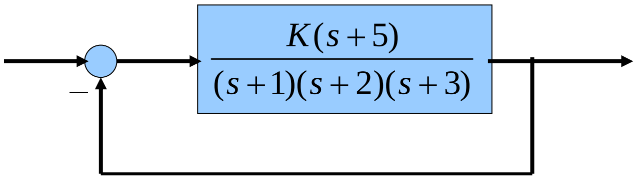
$$P(s)=\frac{s+1}{s+2},\qquad n=m=1$$





Example 9

- Draw root locus for the following system



Characteristic Equation:

$$1+K\frac{s+5}{(s+1)(s+2)(s+3)}=0$$

$$P(s)=\frac{s+5}{(s+1)(s+2)(s+3)}, \qquad n=3, m=1$$

Example 9

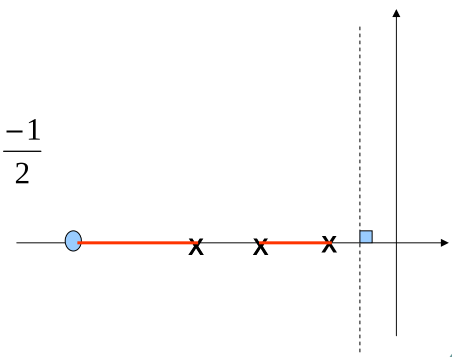
$$P(s)=\frac{s+5}{(s+1)(s+2)(s+3)}, \qquad n=3, m=1$$

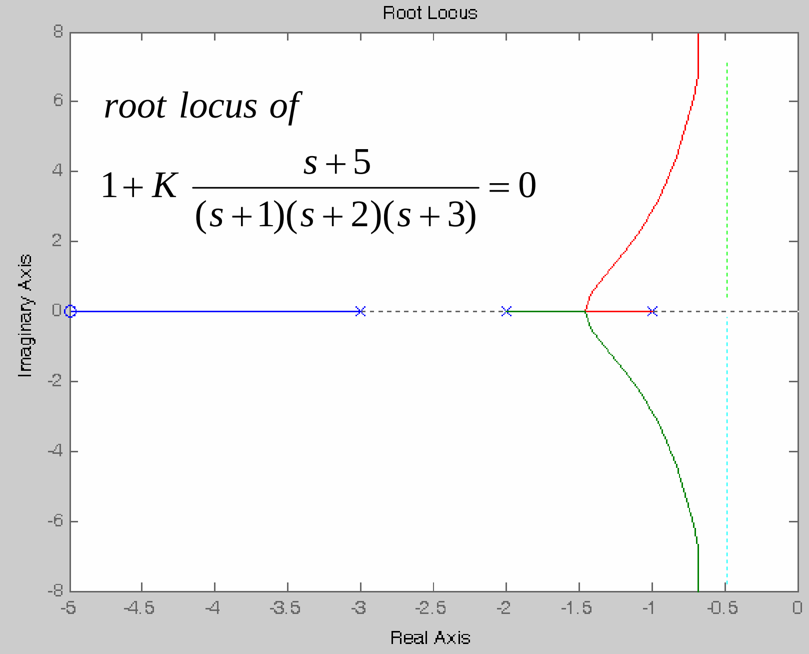
$Poles: -1,-2,-3, \quad Zeros: -5$

$\#of\ symptotes = 3-1=2$

$Angle\ \ 90,\ 270$

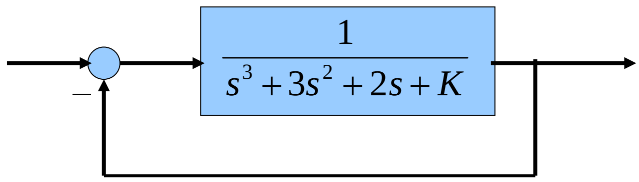
$$Centroid\ = \frac{-1-2-3-(-5)}{3-1}=\frac{-1}{2}$$





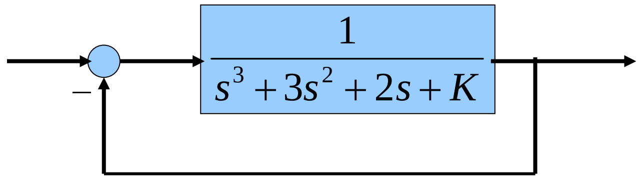
Example10

- How do we draw root locus for this system?



Example10

- How do we draw root locus for this system?



Initial Step: Express the characteristics as

$$1 + K\ P(s) = 0$$

$$What\ is\ P(s)?$$

Example10

Initial Step: Express the characteristics as

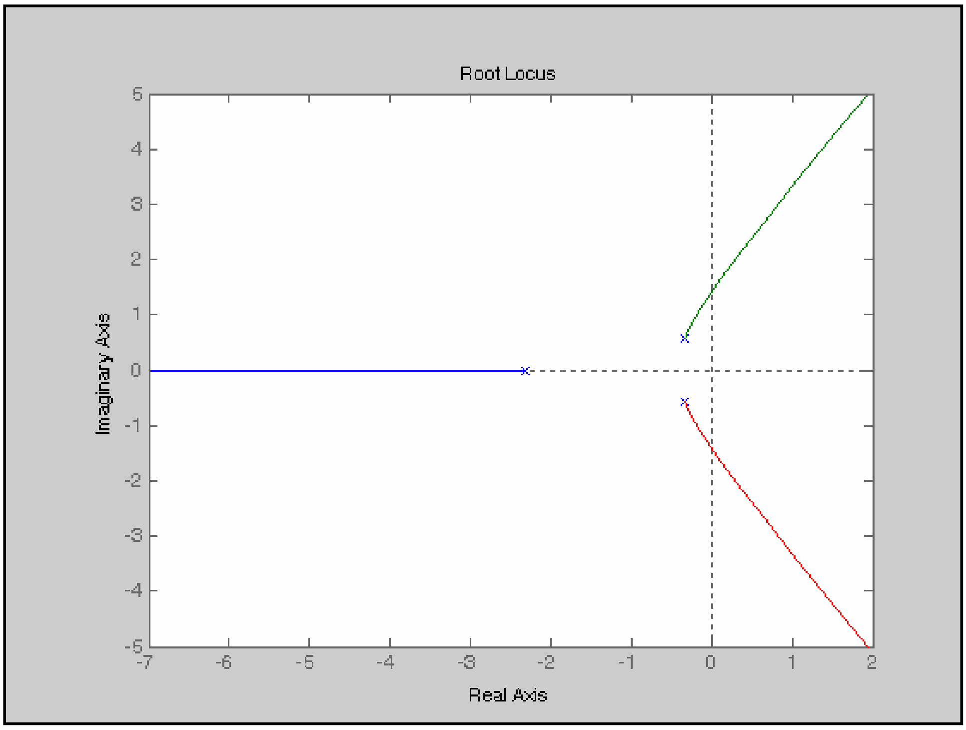
$$\text{Characteristic Equation}:\ 1 + \frac{1}{s^3+3s^2+2s+K} = 0$$

$$s^3+3s^2+2s+K+1 = 0\ = \left(s^3+3s^2+2s+1\right) + K$$

$$1 + K\frac{1}{s^3+3s^2+2s+1} = 0$$

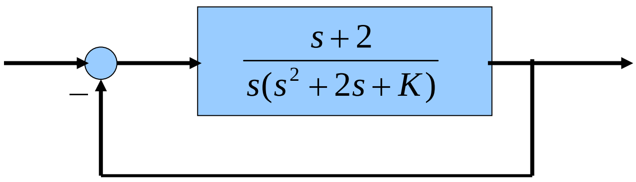
$$\Rightarrow\ P(s) = \frac{1}{s^3+3s^2+2s+1}$$

Continue the Root Locus Procedure



Example11

- How do we draw root locus for this system?



Example11

Initial Step: Express the characteristics as

$$\text{Characteristic Equation}: \ 1+\frac{s+2}{s(s^2+2s+K)}=0$$

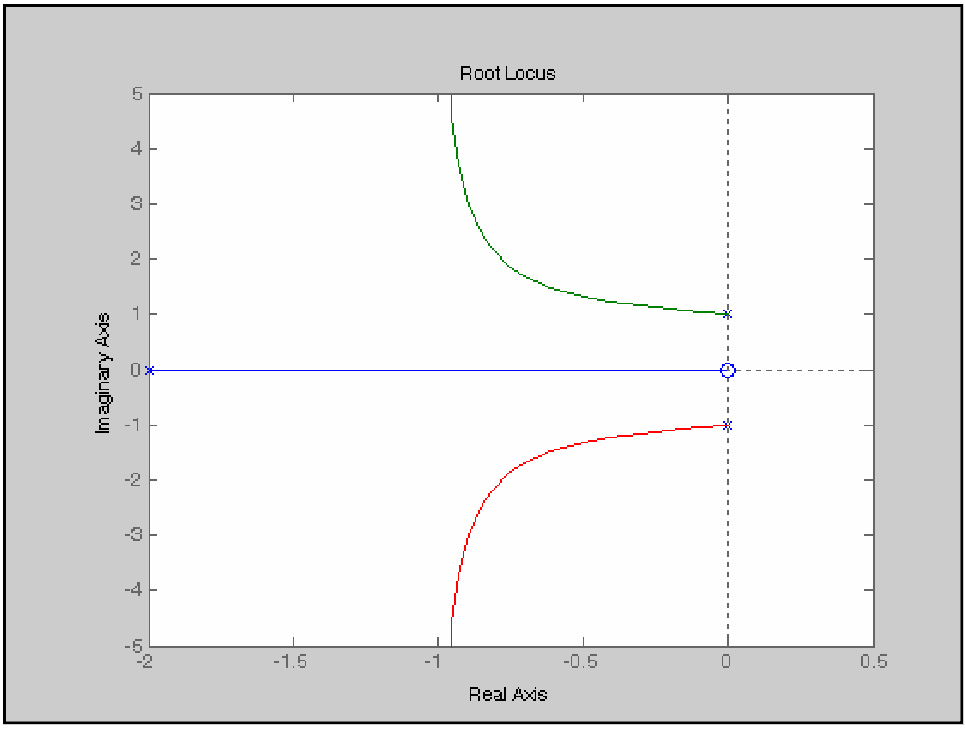
$$s(s^2+2s+K)+s+2=0=s^3+2s^2+Ks+s+2$$

$$\left(s^3+2s^2+s+2\right)+Ks$$

$$1+K\frac{s}{s^3+2s^2+s+2}=0$$

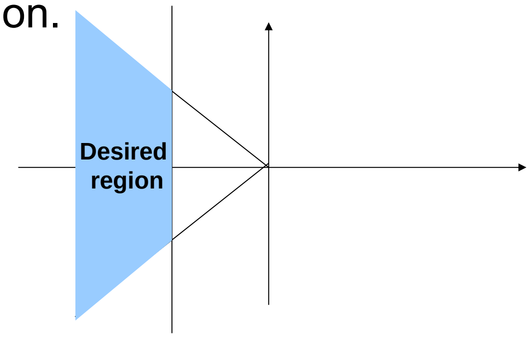
$$\Rightarrow \quad P(s)=\frac{s}{s^3+2s^2+s+2}$$

Continue the Root Locus Procedure

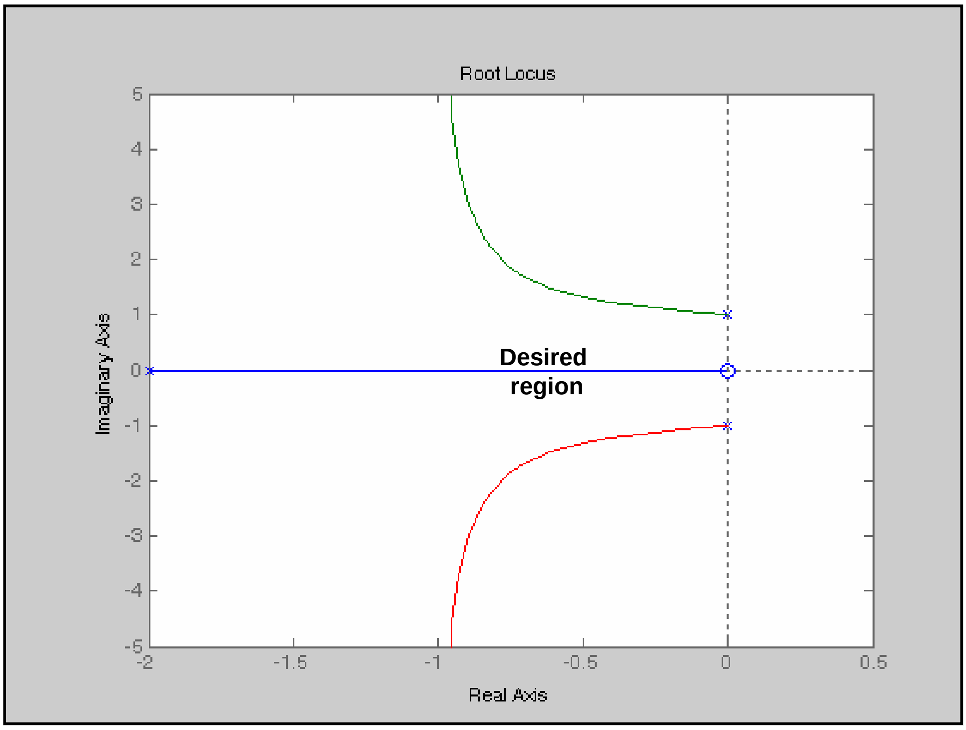


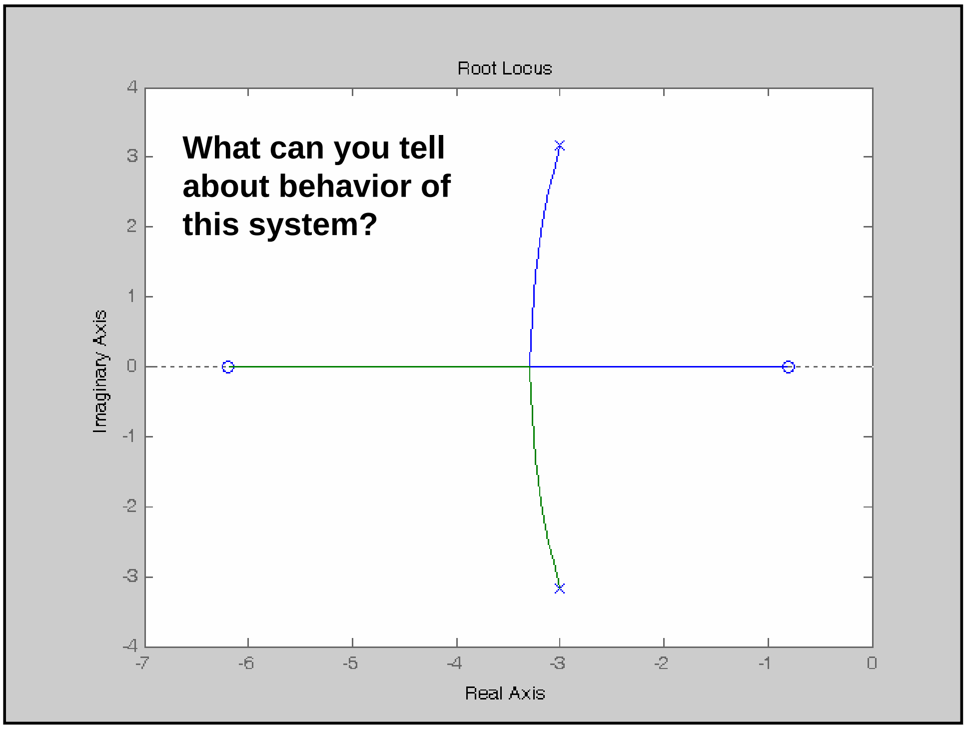
Controller Design Based on Root Locus

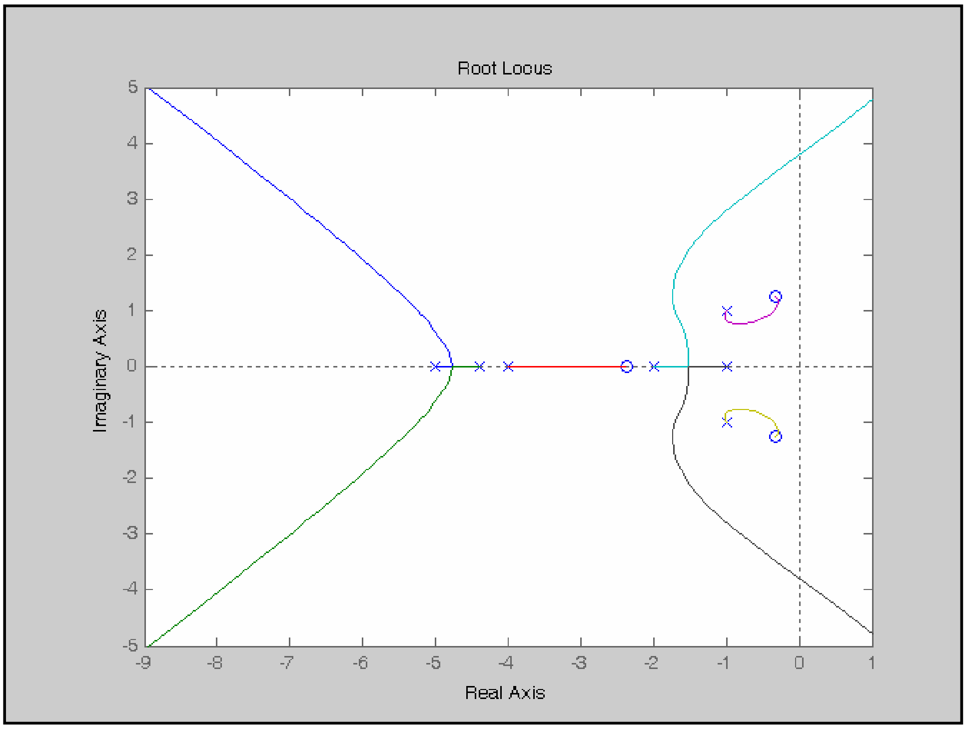
- The idea here is to select the parameter K so that all the poles in the desired location.

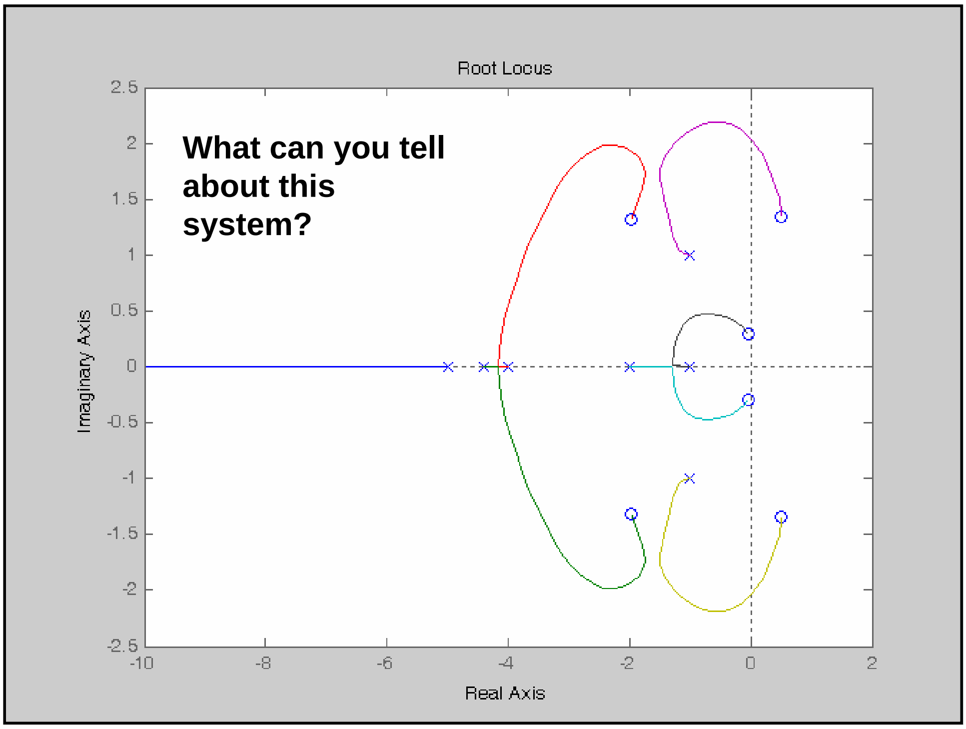


- The desired region is obtained to satisfy the given specifications
- Draw the desired region on the root locus
- Select the gain K such that all poles of the closed loop system are in the desired region


Root Locus
Desired
region
Imaginary Axis
Real Axis
5
4
3
2
1
0
-1
-2
-3
-4
-5
-2
-1.5
-1
-0.5
0
0.5


Root Locus
What can you tell
about behavior of
this system?
Imaginary Axis
Real Axis
4
3
2
1
0
-1
-2
-3
-4
-7
-6
-5
-4
-3
-2
-1
0


Root Locus
Imaginary Axis
Real Axis


Root Locus
What can you tell about this system?
Imaginary Axis
Real Axis

Keywords

- Root locus
- Breakaway points
- Asymptotes
- Centroid
- Angle of departure
- Angle of arrival