Term 081

CISE302: Linear Control Systems

5. Laplace Transform Properties

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Reading Assignment : Section

Learning Objectives

- To be able to state different Laplace transform properties
- To be able to apply different properties to simplify calculations of Laplace transform or Inverse Laplace transform

Definition of Laplace Transform

$$F(s) = L\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt$$

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}ds$$

Linear Properties of Laplace Transform

$$L\{a\, f(t) + b\, g(t)\} = a\, L\{f(t)\} + b\, L\{g(t)\}$$

Proof :

$$L\{a\, f(t) + b\, g(t)\} = \int_0^\infty \left(a\, f(t) + b\, g(t)\right) e^{-st} dt = \int_0^\infty f(t) e^{-st} dt + \int_0^\infty g(t) e^{-st} dt$$

$$= \int_0^\infty a\, f(t) e^{-st} dt + \int_0^\infty b\, g(t) e^{-st} dt$$

$$= a \int_0^\infty f(t) e^{-st} dt + b \int_0^\infty g(t) e^{-st} dt$$

$$= a\, L\{f(t)\} + b\, L\{g(t)\}$$

Linear Properties of Laplace Transform

$$L\{a\,f(t)+b\,g(t)\}=a\,L\{f(t)\}+b\,L\{g(t)\}$$

Special Cases:

Multiplication by constant

$$L\{a\,f(t)\}=a\,L\{f(t)\}$$

Addition of two functions

$$L\{f(t)+g(t)\}=L\{f(t)\}+L\{g(t)\}$$

Multiplication by Exponential

$$Let \;\; L\left\{ f(t) \right\} = F(s) \;\; then$$

$$L\left\{ f(t)\, e^{-at} \right\} = F(s+a)$$

Proof :

$$L\left\{ f(t)\, e^{-at} \right\} = \int_0^\infty f(t) e^{-at} e^{-st} dt = \int_0^\infty f(t) e^{-(a+s)t} dt = F(s+a)$$

Multiplication by Exponential

Examples

$$L\left\{ f(t)\, e^{-at} \right\} = F(s+a)$$

$$L\{1\} = \frac{1}{s}, \qquad L\left\{ e^{-at} \right\} = \frac{1}{(s+a)}$$

$$L\{t\} = \frac{1}{s^2}, \qquad L\left\{ t\, e^{-at} \right\} = \frac{1}{(s+a)^2}$$

$$L\{\sin(\omega t)\} = \frac{\omega}{s^2 + \omega^2}, \qquad L\left\{ \sin(\omega t)\, e^{-at} \right\} = \frac{\omega}{(s+a)^2 + \omega^2}$$

$$L\{\cos(\omega t)\} = \frac{s}{s^2 + \omega^2}, \qquad L\left\{ \cos(\omega t)\, e^{-at} \right\} = \frac{(s+a)}{(s+a)^2 + \omega^2}$$

Useful Identities

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

$$\cos(\theta) = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

$$\sin(\theta) = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

Euler Identity

Example

Proof of Laplace Transform of sin Function

$$f(t) = \sin(\omega t) \quad \xrightarrow{\text{Laplace Transform}} \quad F(s) = \frac{\omega}{s^2 + \omega^2}$$

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty \sin(\omega t)e^{-st}dt =$$

$$\int_0^\infty \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)e^{-st}dt = \frac{1}{2j}\left(\int_0^\infty e^{j\omega t - st}dt - \int_0^\infty e^{-j\omega t - st}dt\right)$$

$$\frac{1}{2j}\left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right) = \frac{\omega}{s^2 + \omega^2}$$

Example

cosine Function

$f(t) = \cos(\omega t)$ → Laplace Transform → $F(s) = \dfrac{s}{s^2 + \omega^2}$

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty \cos(\omega t)e^{-st}dt$$

$$\int_0^\infty \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)e^{-st}dt = \frac{1}{2}\left(\int_0^\infty e^{j\omega t}e^{-st}dt + \int_0^\infty e^{-j\omega t}e^{-st}dt\right)$$

$$= \frac{1}{2}\left(\frac{1}{s + j\omega} + \frac{1}{s - j\omega}\right) = \frac{s}{s^2 + \omega^2}$$

Multiplication by time

$$L\{\, t\, f(t)\} = -\frac{d}{ds}F(s)$$

$$L\{\, u(t)\} = \frac{1}{s}$$

$$L\{\, t\,u(t)\} = -\frac{d}{ds}\left\{\frac{1}{s}\right\} = \frac{1}{s^2}$$

$$L\{\, t^2\, u(t)\} = -\frac{d}{ds}\left\{\frac{1}{s^2}\right\} = \frac{2}{s^3}$$

$$L\{\, t^3\, u(t)\} = -\frac{d}{ds}\left\{\frac{2}{s^3}\right\} = \frac{6}{s^4}$$

$$\Rightarrow \quad L\{\, t^n\, u(t)\} = \frac{n!}{s^{n+1}}$$

Laplace Transform of Derivative

$$L\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0)$$

$$L\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2F(s) - sf(0) - \dot{f}(0)$$

$$L\left\{\frac{d^3 f(t)}{dt^3}\right\} = s^3F(s) - s^2 f(0) - s\dot{f}(0) - \ddot{f}(0)$$

Laplace Transform of Derivative

Example

Apply Laplace trasform to the ODE

$$\ddot{x}(t) + 3\dot{x}(t) + 2x(t) = 1, \;\; x(0) = 4, \dot{x}(0) = 5$$

$$L\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0)$$

$$L\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2F(s) - sf(0) - \dot{f}(0)$$

Solution :

$$L\left\{\ddot{x}(t) + 3\dot{x}(t) + 2x(t)\right\} = L\left\{u(t)\right\}$$

$$\left[s^2X(s) - sx(0) - \dot{x}(0)\right] + 3[sX(s) - x(0)] + 2X(s) = \frac{1}{s}$$

$$\left[s^2X(s) - 4s - 5\right] + 3[sX(s) - 4] + 2X(s) = \frac{1}{s}$$

Laplace Transform of Integrals

$$L\left\{\int_0^t f(\lambda)d\lambda\right\} = \frac{1}{s}F(s)$$

$$Example: \quad L\left\{\int_0^t e^{-2z}\sin(z)dz\right\} = ?$$

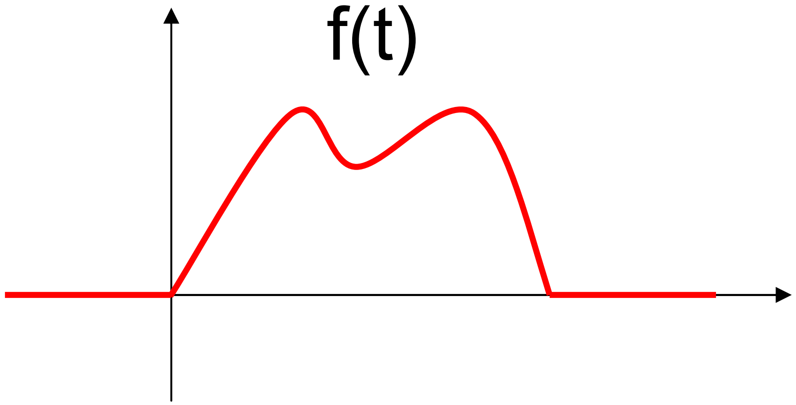
$$L\left\{e^{-2t}\sin(t)\right\} = \frac{1}{(s+2)^2+1}$$

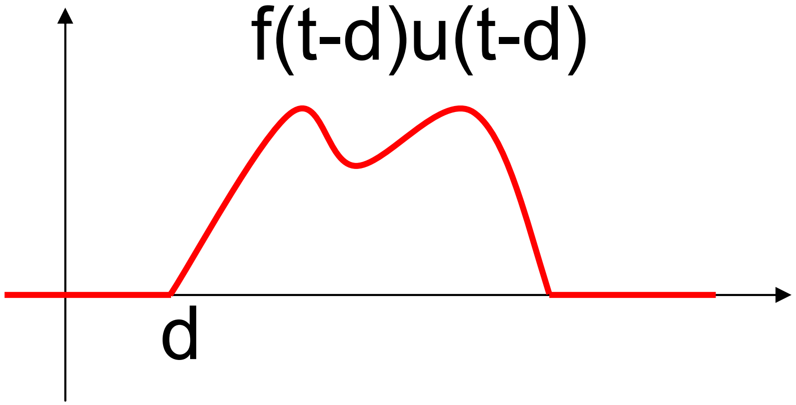
$$L\left\{\int_0^t e^{-2z}\sin(z)dz\right\} = \frac{1}{s((s+2)^2+1)}$$

Laplace Transform of Functions with Delay

$$Let\ \ L\{f(t)\} = F(s)\ \ Then$$

$$L\{f(t-d)u(t-d)\} = e^{-sd}F(s)$$

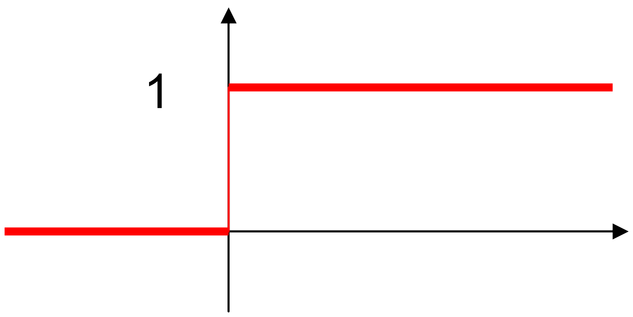


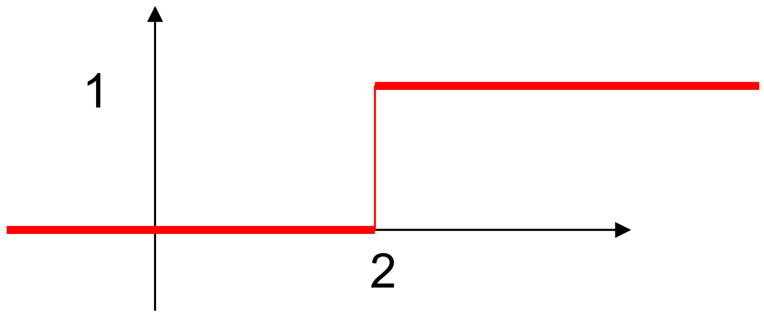


$$F(s) \qquad\qquad e^{-sd}F(s)$$

Laplace Transform of Functions with Delay

Example





$$u(t)$$

$$\frac{1}{s}$$

$$u(t-2)$$

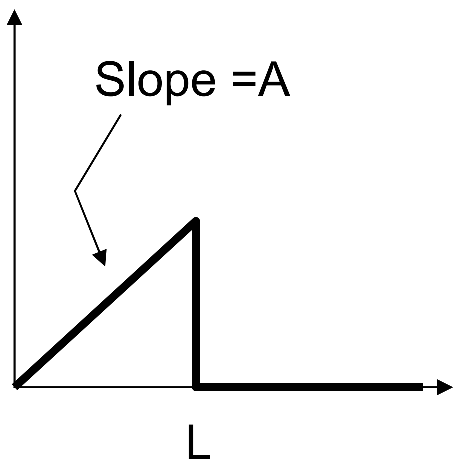
$$\frac{1}{s}e^{-2s}$$

Properties of Laplace Transform

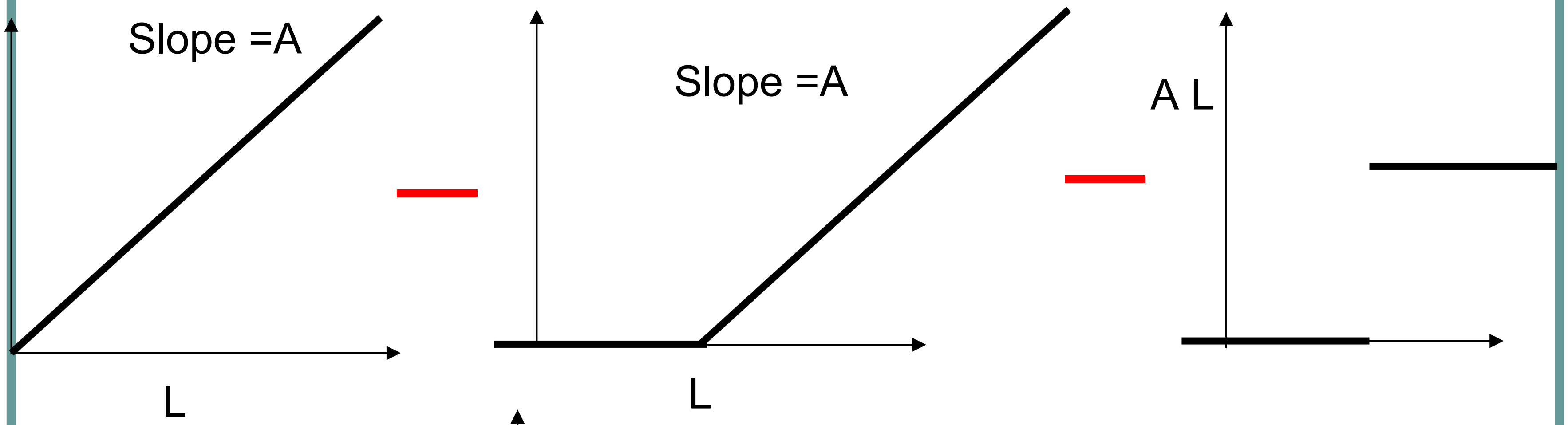
$$f(t) = \begin{cases} 0 & t < 0 \\ At & 0 \le t \le L \\ 0 & t > L \end{cases}$$

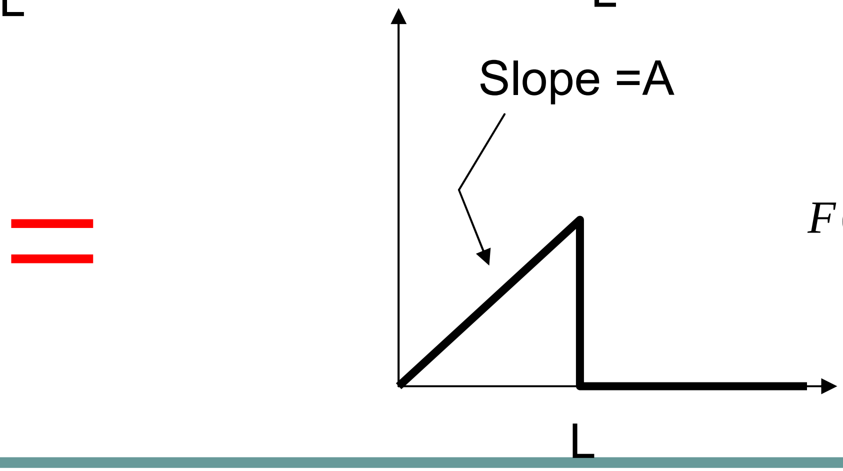
$$f(t) = At\,u(t) - A(t-L)u(t-L) - ALu(t-L)$$

$$F(s) = \frac{A}{s^2} \qquad - A\frac{1}{s^2}e^{-Ls} \qquad - AL\frac{1}{s}e^{-Ls}$$



Properties of Laplace Transform





$$F(s)=\frac{A}{s^2}-A\frac{1}{s^2}e^{-Ls}-AL\frac{1}{s}e^{-Ls}$$

Properties of Laplace Transform

$$L\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0)$$

$$L\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2 F(s) - sf(0) - \dot{f}(0)$$

$$L\left\{\frac{d^3 f(t)}{dt^3}\right\} = s^3 F(s) - s^2 f(0) - s\dot{f}(0) - \ddot{f}(0)$$

$$L\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0) - \dots - s\, f^{(n-2)}(0) - f^{(n-1)}(0)$$

These are essential in solving differential equations

Initial Value& Final Value Theorems

Initial Value $f(0+)$ *and* Final Value $f(\infty)$

can be obtained directly from F(s)

with out the need to obtain

inverse Laplace Transfrom

Initial Value Theorem

$$f(0+) = \lim_{s \to \infty} s\, F(s)$$

the value of the function at the initial time is obtained by taking the limit $\lim_{s \to \infty} s\, F(s)$

If F(s) is not strictly proper then initial condition will be unbounded due to presence of impulse function

Final Value Theorems

$$f(\infty) = \lim_{s \to 0} \; s \, F(s)$$

provided F(s) has no poles on the in the right half of the complex plane and with a possible exception of single pole at the origin.

Examples:

$$\mathrm{G(s)} = \frac{\mathrm{s+5}}{\mathrm{(s+2)(s-3)}}, \quad F(s) = \frac{\mathrm{s+4}}{\mathrm{s(s+2)(s+3)}}$$

We can obtain $f(\infty)$ *but not* $g(\infty)$.

Final value theorem

$$F(s) = \frac{2}{(s+1)(s+4)} = \frac{\frac{2}{3}}{(s+1)} + \frac{-\frac{2}{3}}{(s+4)}$$

$$f(t) = \frac{2}{3}e^{-t} - \frac{2}{3}e^{-4t} \qquad for\ t \geq 0$$

$$f(\infty) = \frac{2}{3}e^{-\infty} - \frac{2}{3}e^{-4\infty} = 0$$

$$f(\infty) = \lim_{s \to 0} \frac{2s}{(s+1)(s+4)} = \frac{0}{(0+1)(0+4)} = 0$$

Final value theorem

$$F(s)=\frac{2}{(s-1)(s+4)}=\frac{\frac{2}{5}}{(s-1)}+\frac{-\frac{2}{5}}{(s+4)}$$

$$f(t)=\frac{2}{5}e^{t}-\frac{2}{5}e^{-4t}\qquad for\ t\geq 0$$

$$f(\infty)=\frac{2}{5}e^{\infty}-\frac{2}{5}e^{-4\infty}=\infty$$

$$f(\infty)=\lim_{s\to 0}\frac{2s}{(s-1)(s+4)}=\frac{0}{(0-1)(0+4)}=0$$

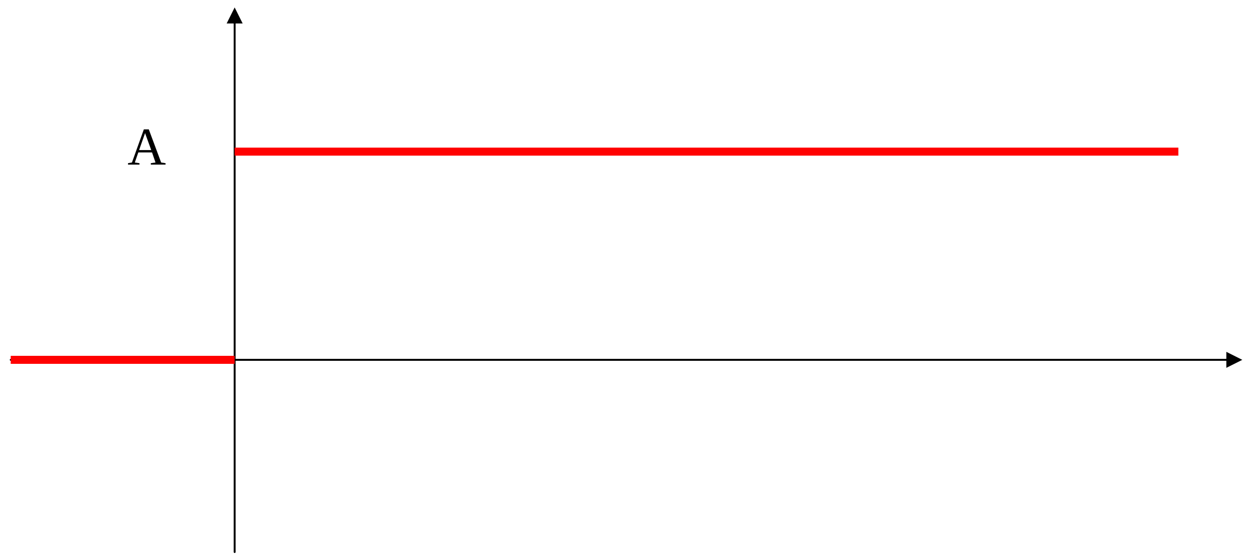
Not valid

Remember we can apply final value theorem if all poles of sF(s) have negative real parts

Step function

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$U(s) = \frac{A}{s}$$



impulse function

$$\delta(t) = 0 \quad for \ t \neq 0$$

$$F(s) = 1$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(t)\, dt = 1, \qquad \varepsilon > 0$$

impulse function

$$\delta(t) = 0 \quad for\ t \neq 0$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(t)\, dt = 1, \ \ \varepsilon > 0$$

$$L\{\delta(t)\} = 1$$

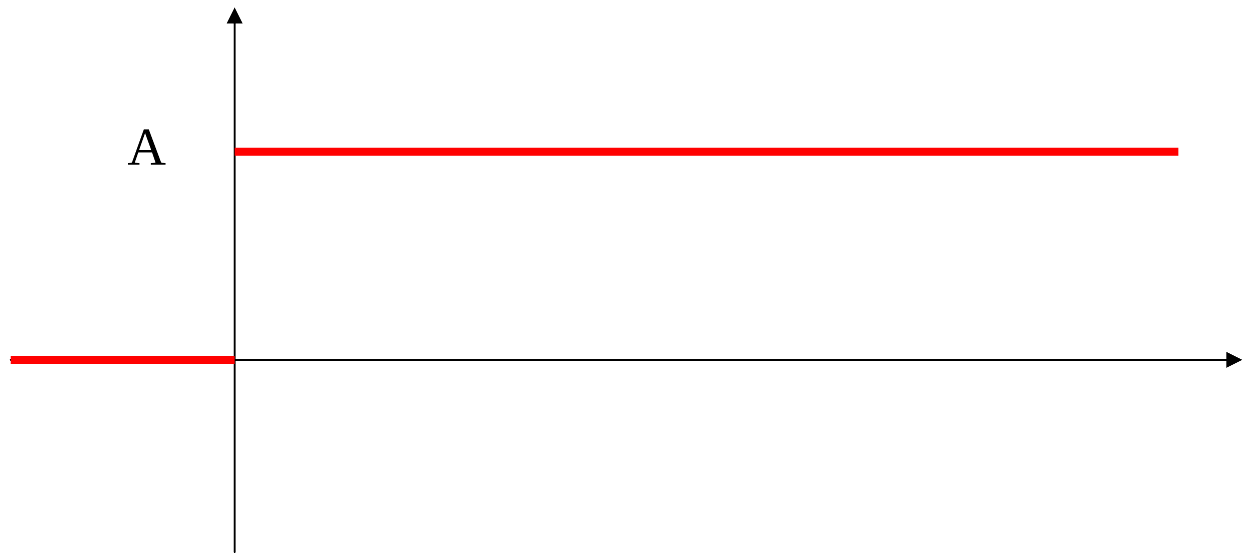
$$\int_{a}^{b} \delta(t-c)\ f(t)dt = \begin{cases} f(c) & if\ \ c \in [a,b] \\ 0 & otherwise \end{cases}$$

sampling property

Step function

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$U(s) = \frac{A}{s}$$



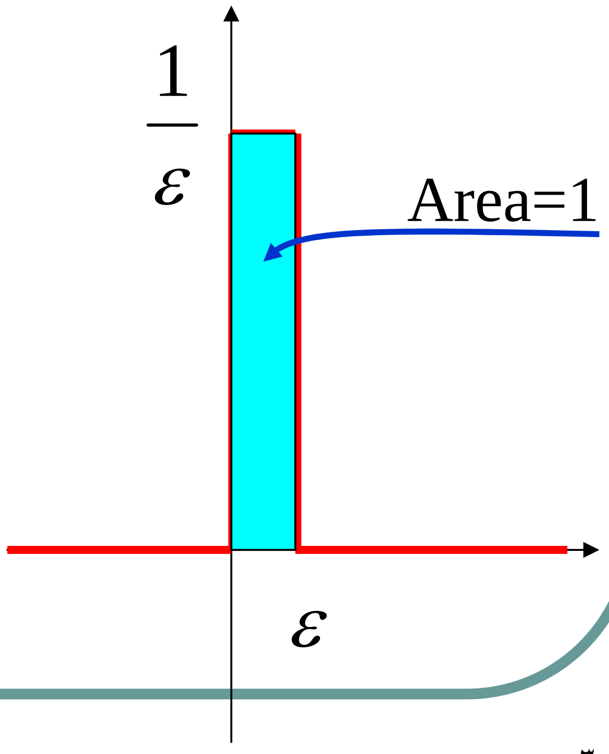
impulse function

$$\delta(t) = 0 \quad for \ t \neq 0 \qquad\qquad F(s) = \quad 1$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(t)\, dt = 1, \qquad \varepsilon > 0$$

impulse function

You can consider the unit impulse as the limiting case for a rectangle pulse with unit area as the width of the pulse approaches zero



impulse function

$$\delta(t) = 0 \quad for \ t \neq 0$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(t)\, dt = 1, \ \ \varepsilon > 0$$

$$L\{\delta(t)\} = \quad 1$$

sampling property

$$\int_{a}^{b} \delta(t-c)\ f(t)dt = \begin{cases} f(c) & if \ \ c \in [a,b] \\ 0 & otherwise \end{cases}$$

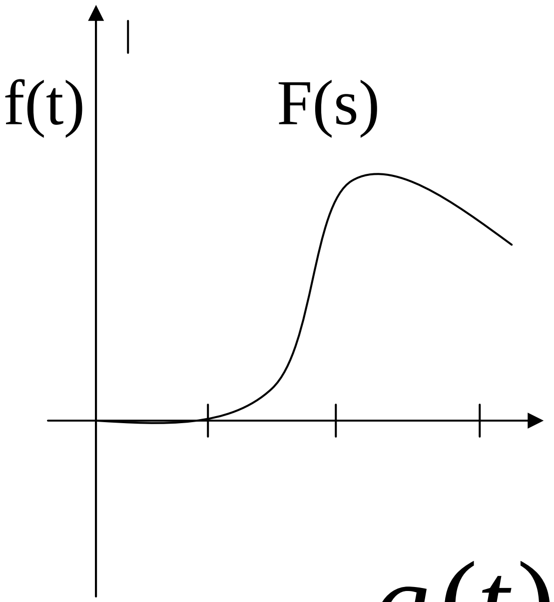
Sample property of impulse function

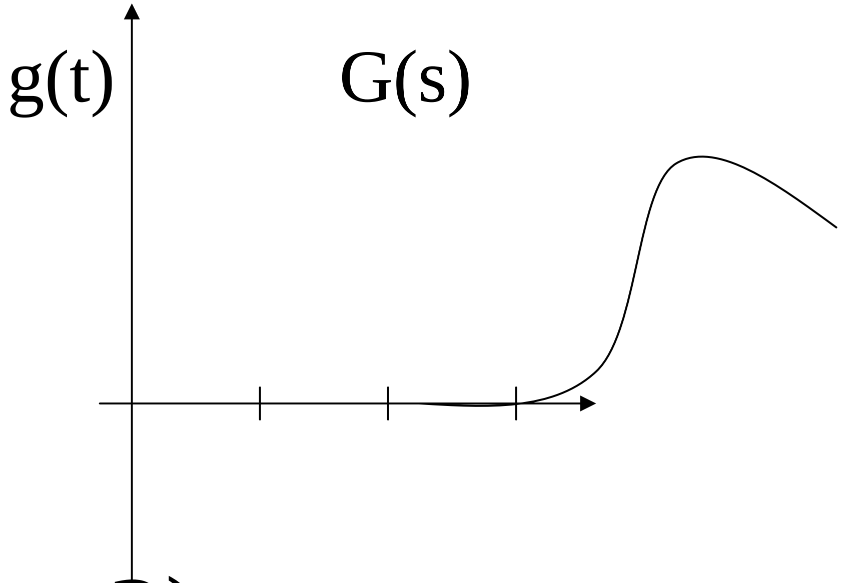
$$\int_{1}^{5} \delta(t-2)\cos(3t)\,dt = \cos(6)$$

$$\int_{1}^{5} \delta(t-3)\,e^{-t}\,dt = e^{-3}$$

$$\int_{-5}^{2} \delta(t-3)\,e^{-t}\,dt = 0$$

Time delay





$$g(t) = f(t-2)$$

$$G(s) = F(s)e^{-2s}$$

Summary

- apply different Laplace transform properties to simplify calculations of Laplace transform or Inverse Laplace transform