SE311: Design of Digital Systems

Lecture 5: Canonical and standard forms of Boolean functions

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(Term 041)

Outlines

- Review of Important Theorems
- Boolean Functions
- Canonical and Standard Forms
- Other Logic Operations

Boolean Algebra
Postulates

$$x + 0 = x \qquad x \bullet 1 = x$$

$$x + x' = 1 \qquad x \bullet x' = 0$$

$$x + y = y + x \qquad x \bullet y = y \bullet x$$

$$x \bullet (y + z) = x \bullet y + x \bullet z \qquad x + (y \bullet z) = (x + y) \bullet (x + z)$$

Basic Theorems of Boolean Algebra

$$x + x = x \qquad x \bullet x = x$$

$$x + 1 = 1 \qquad x \bullet 0 = 0$$

$$(x')' = x$$

$$x + (y + z) = (x + y) + z \qquad x \bullet (y \bullet z) = (x \bullet y) \bullet z$$

$$(x + y)' = x' \ y' \qquad (x \bullet y)' = x' + y'$$

$$x + x \bullet y = x \qquad x \bullet (x + y) = x$$

Generalations of DeMorgan Theorem

$$(A+B+C)' = A'\ B'\ C'$$

$$(A+B+C+D+....+Z)' = A'\ B'\ C' \cdots Z'$$

$$(ABCD....Z)' = A' + B' + C' + \cdots + Z'$$

Proof of $(A+B+C)' = A'\ B'\ C'$

$Let\ \ x = B + C \quad \Rightarrow (A+B+C)' = (A+x)'$

DeMorgan Theorem $\Rightarrow (A+B+C)' = A'\,x'$

$$= A'(B+C)' = A'\ B'\ C'$$

Boolean Function

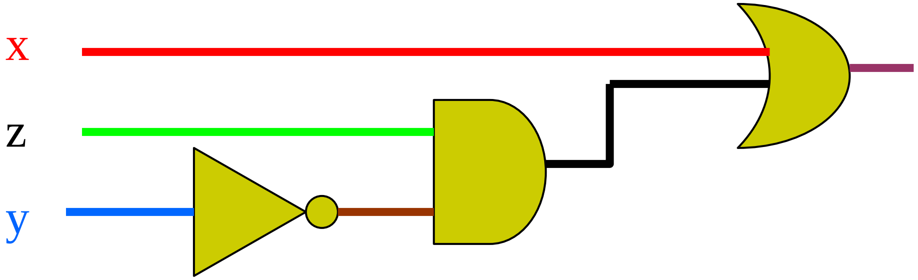
- A Boolean function is described by an expression consisting of binary variables, the constants 0 and 1 and logic operation symbols
- Operator Precedence:
  - Parenthesis (should be evaluated first)
  - NOT
  - AND
  - OR
- A Boolean function can also be represented in a truth table

Boolean Functions

- $F = x + y' z$

Truth Table of F

x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1



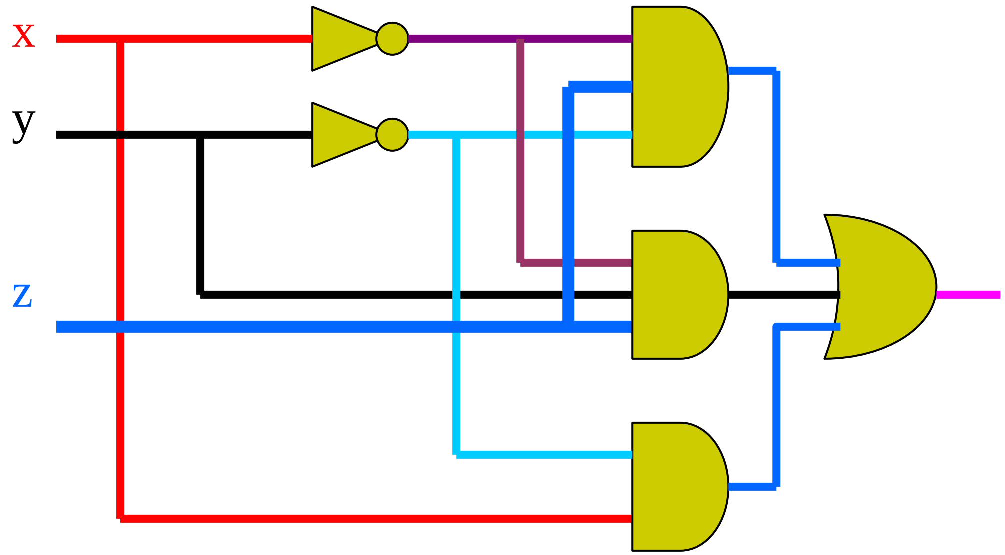
Gate implementation of F

There is a unique truth table for each function but expressions and gate implementations are not unique.

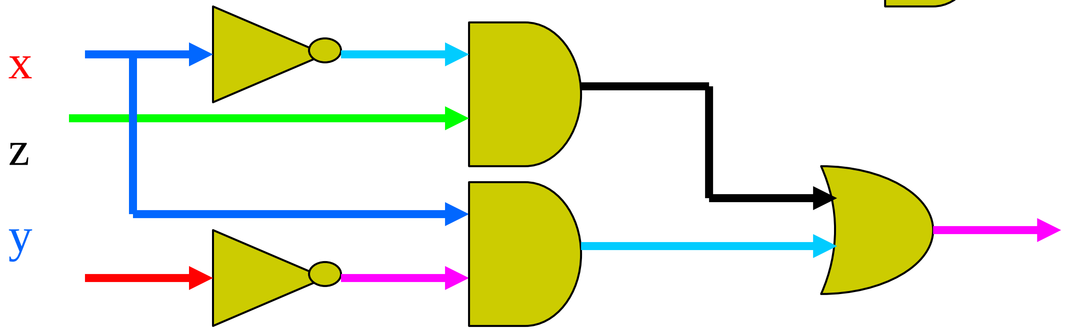
Boolean Functions

Different expressions and implementations of the same function

$$F = x'y'z + x'yz + xy'$$



$$F = x'z + xy'$$



Boolean Functions

$$F = x'z + xy'$$

$$F \ is \begin{cases} 1 & if \ \ (x=0 \ AND \ z=1) \ \ or \ \ (x=1 AND \ \ y=0) \\ 0 & \qquad\qquad otherwise \end{cases}$$

$$F \ is \begin{cases} 1 & if \ \ (xz=01) \ \ or \ \ (xy=10) \\ 0 & \qquad otherwise \end{cases}$$

Complement of a function

F' : the complement of F

Two methods to compute the complement

1. Use DeMorgan Theorem

   $(A+B+C)' = A'B'C'$

   $(ABC)' = A'+B'+C'$

2. Obtain the Dual of F then complement each literal

   $0 \leftrightarrow 1$, AND $\leftrightarrow$ OR then complement

Complement of a function

Example

$$F = x'yz' + x'y'z$$

Method 1:

$$F' = (x'yz' + x'y'z)' = (x'yz')'(x'y'z)'$$
$$= (x + y' + z)(x + y + z')$$

Method 2:

$$\text{Dual of } F = (x' + y + z')(x' + y' + z)$$
$$F' = (x + y' + z)(x + y + z')$$

Canonical and Standard Forms

- A binary variable may appear in
  - Normal form x
  - Complemented form x'
- With two binary variables 4 possible combinations of AND terms x'y', x'y, xy', and xy. Each one is called a **minterm**
- With two binary variables 4 possible combinations of OR terms (x'+y'), (x+y'), (x'+y), and (x+y). Each one is called a **maxterm**

Minterms and Maxterms for two binary variables

		Minterms		Maxterms	
x	y	*term*	*Designation*	*term*	*Designation*
0	0	$x'y'$	m_0	$(x+y)$	M_0
0	1	$x'y$	m_1	$(x+y')$	M_1
1	0	xy'	m_2	$(x'+y)$	M_2
1	1	xy	m_3	$(x'+y')$	M_3

Minterms and Maxterms for two binary variables

$$Question: \ What \ do \ we \ call \ these$$

$x'y'$	m_0
$(x + y')$	M_1
$(x' + y)$	M_2
xy	m_3

Minterms and Maxterms for three binary variables

x	y	z	Minterms: term	Minterms: Designation	*Maxterms*: term	*Maxterms*: Designation
0	0	0	$x'y'z'$	m_0	$(x+y+z)$	M_0
0	0	1	$x'y'z$	m_1	$(x+y+z')$	M_1
0	1	0	$x'yz'$	m_2	$(x+y'+z)$	M_2
0	1	1	$x'yz$	m_3	$(x+y'+z')$	M_3
1	0	0	$xy'z'$	m_4	$(x'+y+z)$	M_4
1	0	1	$xy'z$	m_5	$(x'+y+z')$	M_5
1	1	0	xyz'	m_6	$(x'+y'+z)$	M_6
1	1	1	xyz	m_7	$(x'+y'+z')$	M_7

Minterms and Maxterms for three binary variables

			Minterms		Maxterms	
x	y	z	term	Designation	term	Designation
1	0	0	$x\,y'\,z'$	m_4	$(x'+y+z)$	M_4

$$m_4 = x\,y'\,z' = 1 \quad if \quad x = 1\ AND \quad y = 0\ AND \quad z = 0$$

$$M_4 = 0 \quad if \quad x = 1\ AND \quad y = 0\ AND \quad z = 0$$

$$Note\,that \quad M_k = m_k'$$

Canonical Forms

- Any Boolean function can be expressed as the sum (OR) of **minterms.**
- Any Boolean function can be expressed as the product of (AND) of **maxterms.**
- With n variables there are 2^n minterms. Each minterms contains exactly n literals (the variables in normal or complemented forms)

Canonical Forms

- If a term contains less than n literals we can express it as sum (OR) of minterms. We can do this by ANDing it with terms like (x+x’)

In two variable case A = A(B+B’)=AB+AB’

Canonical Forms

Example: sum of minterms

$Write\ \ F = A + B'C$ as sum of minterms

$$A = A(B + B') = AB + AB'$$

$$= AB(C + C') + AB'(C + C')$$

$$= ABC + ABC' + AB'C + AB'C'$$

$$B'C = (A + A')B'C = AB'C + A'B'C$$

$$F = ABC + ABC' + \boxed{AB'C} + AB'C' + \boxed{AB'C} + A'B'C$$

$$F = ABC + ABC' + AB'C + AB'C' + A'B'C$$

$$F = m_7 + m_6 + m_5 + m_4 + m_1$$

$$F(A, B, C) = \sum(1,4,5,6,7)$$

Boolean Function

x	y	z	$F(x,y,z)$		
0	0	0	0		
0	0	1	1	$x'y'z$	m_1
0	1	0	0		
0	1	1	1	$x'yz$	m_3
1	0	0	0		
1	0	1	0		
1	1	0	0		
1	1	1	0		

$$F = \sum(1,3) = x'y'z + x'yz = x'z$$

Boolean Functions

x	y	z	$F(x,y,z)$	
0	0	0	0	
0	0	1	1	M_1
0	1	0	0	
0	1	1	1	M_3
1	0	0	0	
1	0	1	0	
1	1	0	0	
1	1	1	0	

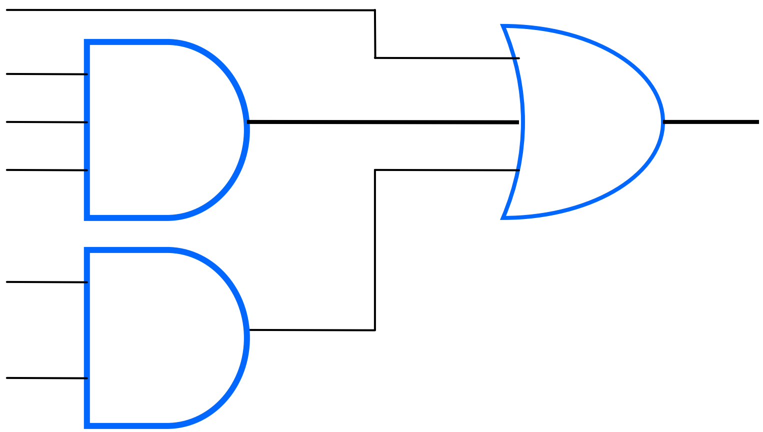
$$F = \prod(0,2,4,5,6,7) = M_0\, M_2\, M_4\, M_5\, M_6\, M_7$$

$$F = (x+y+z)(x+y'+z)(x'+y+z)(x'+y+z')(x'+y+z')(x'+y'+z)(x'+y'+z')$$

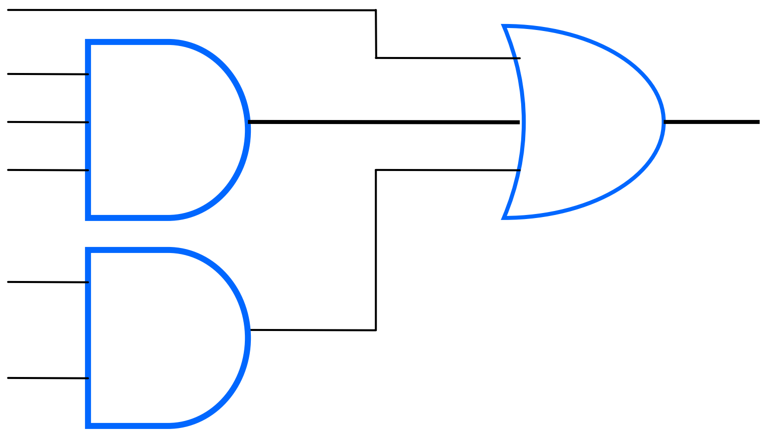
Standard Forms

- Sum of Product
  - $F = x + yx' + y'z$
  - Can be implemented by AND gates followed by an OR gate
- Product of sums
  - $F_2 = (x + y + z)(x' + y + z')$
  - Can be implemented by OR gates followed by an AND gate

Boolean Functions



Sum of Products



Product of Sums

Summary

- Boolean functions
- Canonical Forms
  - Sum of Minterms
  - Product of Maxterms
- Standard forms
  - Product of sums
  - Sum of Product