SE311: Design of Digital Systems

Lecture 4: Boolean Algebra

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Outlines

- Basic Definitions
- Huntington Postulates
- Boolean Algebra
- Basic Postulates and Theorems

Basic Definitions

- Set: Collection of objects having common properties
- Binary Operator (on a set S) : a rule that assigns to each pair of elements of S a unique element from S.

  $*$ is binary operator on S if for all $a,b \in \mathrm{S}$, $a*b = c$, $c \in \mathrm{S}$

  Example : S={0,1}, * is binary operator defined as

  0*0=0; 0*1=0; 1*0=0; 1*1=1

- Closure:

  A set S is closed with respect to the binary operator $*$

  if for all $a,b \in \mathrm{S}$, $a*b = c$, $c \in \mathrm{S}$

Basic Definitions

Commutative:

The binary operator $*$ is commutative

if $a*b = b*a \quad for\ all\ a,b \in \mathrm{S}$

Identity Element:

$e \in S$ is identity element w.r.t. $*$ if $x*e = e*x = x \ \forall x \in S$

Inverse:

A set S having identity element e w.r.t. the binary operator $*$

is said to have inverse if for every $x \in S \ \exists y \in S$ such that $x*y = e$

Distributive Law:

If $*$ and . are two binary operator on S, $*$ is distibutive over . if

$\mathrm{x} * (\mathrm{y} \,.\, \mathrm{z}) = (\mathrm{x} * \mathrm{y}) \,.\, (\mathrm{x} * \mathrm{z})$

Boolean Algebra
Huntington Postulates

$+$ and $\bullet$ are two binary operator defined on the set B

Huntington Postulates :

1. B is closed w.r.t. $+$, B is closed w.r.t. $\bullet$
2. 0 is identity w.r.t $+$ $(x+0=x)$ 1 is identity w.r.t $\bullet$ $(x \bullet 1 = x)$
3. Commutative $x+y=y+x$ $x \bullet y = y \bullet x$
4. Distributive $x+(y \bullet z)=(x+y)\bullet(x+z)$ $x \bullet (y+z)=(x \bullet y)+(x \bullet z)$
5. For every element $x \in B$ there exist an element $x' \in B$ such that

   $x+x'=1$ *and* $x \bullet x'=0$

6. The exist at least two elements $x, y \in B$ such that $x \neq y$

Boolean Algebra

To have Boolean Algebra, the set B and the two binary operators are defined such that the Huntington postulates are satisfied.

B={0,1}

x	x'
0	1
1	0

x	y	x+y	x•y
0	0	0	0
0	1	1	0
1	0	1	0
1	1	1	1

Boolean Algebra

Comparison with ordinary algebra

- Huntington Postulates do not include associative law but it holds for Boolean algebra
- $x + (y \bullet z) = (x + y) \bullet (x + z)$ is valid for Boolean algebra but not not ordinary algebra
- Boolean algebra do not have additive or multiplicative inverse (subtraction and division are not defined)
- x' The complement of x is not defined for ordinary algebra
- In Boolean algebra B is defined as $\{0,1\}$

Boolean Algebra Verification

- $\text{Closure : From tables the result of operations is 0 or 1}$
- $0 \text{ is identity w.r.t} + \text{ and 1 is identity w.r.t} \bullet$
- Commutative law hold
- $\textit{Both} \quad \mathrm{x} + (\mathrm{y} \bullet z) = (x + y) \bullet (x + z) \text{ and}$

  $\mathrm{x} \bullet (\mathrm{y} + z) = (x \bullet y) + (x \bullet z) \text{ are valid}$
- Conditions on the complement are satisfied
- $\text{B has two elements } \{0,1\}$

Boolean Algebra

Postulates

$$x + 0 = x \qquad x \bullet 1 = x$$

$$x + x' = 1 \qquad x \bullet x' = 0$$

$$x + y = y + x \qquad x \bullet y = y \bullet x$$

$$x \bullet (y + z) = x \bullet y + x \bullet z \qquad x + (y \bullet z) = (x + y) \bullet (x + z)$$

Boolean Algebra
Duality

- Dual expressions: are obtained by replacing 0 by 1, 1 by 0 AND with OR and OR by AND.
- If an expression is valid then its dual is valid

Expression	Dual
$x+0=x$	$x \bullet 1 = x$
$x+x'=1$	$x \bullet x' = 0$
$x+y=y+x$	$x \bullet y = y \bullet x$
$x \bullet (y+z) = x \bullet y + x \bullet z$	$x + (y \bullet z) = (x+y) \bullet (x+z)$

Proving Theorems

- Using the Huntington Postulates
- Using Truth Tables

$Theorem \quad x + x = x$

Proof :

$$x + x = (x + x) \bullet 1$$
$$= (x + x) \bullet (x + x')$$
$$= x + xx' = x + 0 = x$$

x	x+x
0	0
1	1

$$x = x \bullet 1$$
$$x + x' = 1$$
$$x \bullet x' = 0$$

Basic Theorems of Boolean Algebra

$$x + x = x$$

$$x + 1 = 1$$

$$(x')' = x$$ *involution*

$$x + (y + z) = (x + y) + z$$ *associative*

$$(x + y)' = x' \; y'$$ *DeMorgan*

$$x + xy = x$$ *Absorption*

Basic Theorems of Boolean Algebra

Dual Theorems

$x + x = x$ | $x \bullet x = x$

$x + 1 = 1$ | $x \bullet 0 = 0$

$(x')' = x$

$x + (y + z) = (x + y) + z$ | $x \bullet (y \bullet z) = (x \bullet y) \bullet z$

$(x + y)' = x' \; y'$ | $(x \bullet y)' = x' + y'$

$x + x \bullet y = x$ | $x \bullet (x + y) = x$

Theorem 2

$$Theorem\ 2:\ x+1=1$$

Proof :

$$x+1=1\bullet(x+1)$$

$$=(x+x')\bullet(x+1)$$

$$=x+x'\bullet 1$$

$$=x+x'$$

$$=1$$

$$x=1\bullet x$$

$$1=(x+x')$$

$$x=x\bullet x, x=x+x$$

x	x+1
0	1
1	1

Theorem 6

$$Theorem\ 6: \ x + xy = x$$

Proof :

$$x + x \bullet y = 1 \bullet x + x \bullet y$$

$$= x \bullet (1 + y)$$

$$= x \bullet 1$$

$$= x$$

$$x = 1 \bullet x$$

$$x \bullet (1 + y) = x \bullet 1 + x \bullet y$$

$$1 + y = 1$$

x	y	x+xy
0	0	0
0	1	0
1	0	1
1	1	1

DeMorgan Law

$$(x+y)' = x' \ \ y'$$

x	y	x’	y’	(x+y)’	x’y’
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	0	0

Operator Precedence

- In evaluating Boolean Expressions
  - Parenthesis
  - NOT
  - AND
  - OR
- x+y'z