Tests for Matrix Definiteness

These tests apply only to symmetric matrices.

1) Eigenvalues Test:
The eigenvalues of the matrix Q are calculated by solving $|Q - \lambda I| = 0$. The values of λ are called the eigenvalues.

- If all the eigenvalues are positive, then Q is positive definite.
- If some are positive and some are zero, Q is positive semidefinite.
- If some are positive and some are negative, Q is indefinite.

2) Product Test
This test is based on value of the product $x^T Qx$ for some specific values of $x \neq 0$.

- If the product $x^T Qx > 0$, Q is positive definite.
- If $x^T Qx \geq 0$, Q is positive semidefinite.
- If $x^T Qx$ can be positive and negative, Q is indefinite.

3) Determinant Test
This test relies on the computation of the determinants of the leading principal minors of Q.

- If $|q_{11}| > 0, \begin{vmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{vmatrix} > 0, \begin{vmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{vmatrix} > 0, \cdots, \begin{vmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{vmatrix} > 0$, then Q is positive definite.
- If the determinants of some of the principal minors are positive and the rest are zero, Q is positive semidefinite.
- If $|q_{11}| < 0, \begin{vmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{vmatrix} > 0, \begin{vmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{vmatrix} < 0, \cdots$ (alternating in sign), then Q is negative definite.
- If, in addition to the last condition above, the determinants of some of the principal minors are zero, Q is negative semidefinite.

- If none of the above conditions apply, Q is indefinite.

Properties of Definite Matrices
1) If Q is positive (semi) definite, then –Q is negative (semi) definite.
2) If Q is positive (semi) definite, then Q^{-1} is positive (semi) definite.
3) If Q and A are positive definite, then Q+A is positive definite.

Example of Matrix Definiteness

Test $Q = \begin{bmatrix} 4 & -10 \\ -10 & 2 \end{bmatrix}$

(1) Product Test

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & -10 \\ -10 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 4x_1 - 10x_2 & -10x_1 + 2x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 4{x_1}^2 - 10x_1x_2 - 10x_1x_2 + 2{x_2}^2$$

$$= 4{x_1}^2 - 20x_1x_2 + 2{x_2}^2$$

For $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x^T Q x = -14 < 0$

For $x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $x^T Q x = 26 > 0$

Hence, Q is indefinite.

(2) Eigenvalues Test

$$|Q - \lambda I| = \begin{vmatrix} 4-\lambda & -10 \\ -10 & 2-\lambda \end{vmatrix} = 8 - 6\lambda + \lambda^2 - 100 = 0 \rightarrow \lambda = \begin{cases} 13.05 > 0 \\ -7.05 < 0 \end{cases}$$

Hence, Q is indefinite.

(3) Determinant Test

$$|4| = 4 > 0$$

$$\begin{vmatrix} 4 & -10 \\ -10 & 2 \end{vmatrix} = 8 - 100 = -92 < 0$$

Hence, Q is indefinite.

Exercises:

Determine the definiteness of the following matrices:

1. $$\begin{bmatrix} 3 & 1.9 & -0.6 \\ 1.9 & 4 & 5 \\ -0.6 & 5 & 11 \end{bmatrix}$$

2. $$\begin{bmatrix} 6 & -5 & 2 \\ -5 & 4.6 & -9 \\ 2 & -9 & 7 \end{bmatrix}$$

3. $$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$