Augmented Lagrangian Penalty Functions

- They can recover an exact optimum for finite penalty parameter values and they enjoy the property of being differentiable.
- For simplicity, let us begin by discussing the case with only equality constraints.
- Consider Problem P of minimizing f(x) subject to $h_i(x) = 0$ for i = 1, ... , l.
- If we employ the quadratic penalty function, then we typically need to let $\lambda \to \omega$ to obtain a constrained optimum for P.
- Define

$$F_{\text{ALAG}}(\mathbf{x}, \mathbf{v}) = f(\mathbf{x}) + \sum_{i=1}^{l} v_i h_i(\mathbf{x}) + \mu \sum_{i=1}^{l} h_i^2(\mathbf{x})$$

- Observe that if $(\bar{x}, \bar{v})$ is a primal-dual KKT solution for P, then indeed at $v = \bar{v}$

$$\nabla F_{\text{ALAG}}(\bar{\mathbf{x}}, \bar{\mathbf{v}}) = \left[f(\bar{\mathbf{x}}) + \sum_{i=1}^{l} \bar{v}_i \nabla h_i(\bar{\mathbf{x}}) \right] + 2\mu \sum_{i=1}^{l} h_i(\bar{\mathbf{x}}) \nabla h_i(\bar{\mathbf{x}}) = \mathbf{0}$$

- Whereas we needed to take $\mu \rightarrow \omega$ to recover $\bar{x}$ in a limiting sense using the quadratic penalty function, it is possible that we only need to make μ large enough for the critical point $\bar{x}$ of $F_{ALAG}(., \bar{v})$ to be its (local) minimizer.

- Observe that F_{ALAG} is the ordinary Lagrangian function augmented by the quadratic penalty term, hence the name augmented Lagrangian penalty function.
- F_{ALAG} can be viewed as the usual quadratic penalty function with respect to the following problem that is equivalent to P:

$$\text{minimize}\left\{ f(\mathbf{x}) + \sum_{i=1}^{l} v_i h_i(\mathbf{x}) : \sum_{i=1}^{l} h_i\ (\mathbf{x}) = 0 \text{ for } i = 1,2,\ldots,l \right\}$$

or

$$\text{minimize}\left\{ f(\mathbf{x}) + \mu \sum_{i=1}^{l} h_i^2(\mathbf{x}) : \sum_{i=1}^{l} h_i\ (\mathbf{x}) = 0 \text{ for } i = 1,2,\ldots,l \right\}$$

- F_{ALAG} is therefore called a multiplier penalty function.

Theorem

Consider Problem P to minimize $f(x)$ subject to $h_i(x) = 0$ for i = l, . .. , l, and let the KKT solution $(\bar{x}, \bar{v})$ satisfy the second-order sufficiency conditions for a local minimum (the Hessain is positive definite). Then, there exists a $\bar{\mu}$ such that for $\mu \geq \bar{\mu}$, $F_{ALAG}(. , \bar{v})$ also achieves a strict local minimum at $\bar{x}$. If f is convex and h_i are affine, then any minimizing solution $\bar{x}$ for P also minimizes $F_{ALAG}(., \bar{v})$ for all $\mu \geq 0$.

Schema of an Algorithm Using Augmented Lagrangian Functions: Method of Multipliers

- The method of multipliers is an approach for solving nonlinear programming problems by using the augmented Lagrangian penalty function in a manner that combines the algorithmic aspects of both Lagrangian duality methods and penalty function methods.
- To monitor individual constraint violation, we can write

$$F_{\text{ALAG}}(\mathbf{x}, \mathbf{v}) = f(\mathbf{x}) + \sum_{i=1}^{l} v_i h_i(\mathbf{x}) + \sum_{i=1}^{l} \mu_i h_i^2(\mathbf{x})$$

Initialization Select some initial Lagrangian multipliers $\bar{v} = \bar{v}$ and positive values $\mu_1, \ldots, \mu_l$ for the penalty parameters. Let x_o be a null vector, and denote VIOL(x_o) = ω, where for any $x \in E_n$, VIOL(x) = maximum $\{|h_j(x)| : i = 1, \ldots, l\}$ is a measure of constraint violations. Put k = 1 and proceed to the "inner loop" of the algorithm.

Inner Loop: Penalty Function Minimization Solve min $F_{ALAG}(x, v)$ s. t. $x \in E_n$ and let x_k the optimal solution obtained. If VIOL(x_k) = 0, stop with x_k as a KKT point. Otherwise, if VIOL(x_k) ≤ 0.25VIOL(x_{k-1})' proceed to the outer loop. If VIOL(x_k) > $\frac{1}{4}$ VIOL(x_{k-1}) then, for each constraint i = 1, ... , l for which $|h_i(x_k)| > \frac{1}{4}$ VIOL(x_{k-1}), replace the corresponding penalty parameter μ_i by 10 μ_i and repeat this inner loop step.

Outer Loop: Lagrange Multiplier Update Replace $\bar{v}$ by $\bar{v}_{new}$, where

$$(\bar{\mathbf{v}}_{\text{new}})_i = \bar{v}_i + 2\mu_i h_i(\mathbf{x}_k) \qquad i = 1, \ldots, l$$

Increment k by 1, and return to the inner loop.