

# Review of the simplex method

Objectives of the topic:

- Present the simplex method in matrix form.
- See chapter 7 in the textbook

- Before a linear programming problem can be solved using the simplex method, the problem has to be written in a form acceptable to the simplex method.
- The acceptable form is
  - The constraints are all 'equal to' type,
  - All the variables are greater than or equal to zero,
  - All slack starting solution.

- Take this LP problem

$$\begin{array}{ll} \text{Maximize} & z = 5x_1 + 4x_2 \\ \text{subject to} & 6x_1 + 4x_2 \leq 24 \\ & x_1 + 2x_2 \leq 6 \\ & -x_1 + x_2 \leq 1 \\ & x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

- This problem has all 'less than or equal to' constraints, which provide easily an all-slack starting solution.

- The revised problem that can be started with in the simplex method is

$$\begin{array}{llllll}
 \text{Maximize} & z = 5x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4 & & & & \\
 \text{subject to} & 6x_1 + 4x_2 + s_1 & & & & = 24 \\
 & x_1 + 2x_2 & +s_2 & & & = 6 \\
 & -x_1 + x_2 & & +s_3 & & = 1 \\
 & x_2 & & & +s_4 & = 2 \\
 & x_1, x_2, s_1, s_2, s_3, s_4 \geq 0 & & & & 
 \end{array}$$

- The problem can be conveniently presented in a table from, called the tableau.

- The problem put in the tableau is

Basic	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	Solution
z	-5	-4	0	0	0	0	0
$s_1$	6	4	1	0	0	0	24
$s_2$	1	2	0	1	0	0	6
$s_3$	-1	1	0	0	1	0	1
$s_4$	0	1	0	0	0	1	2

- The number of rows in the tableau is equal to the number constraints, plus one row for the objective function.
- The entries in the first row are the coefficients in the objective function with opposite signs.

- The starting solution is

$$x_1 = 0$$

$$x_2 = 0$$

$$s_1 = 24$$

$$s_2 = 6$$

$$s_3 = 1$$

$$s_4 = 2$$

$$z = 0$$

- The columns under the starting basic variables form what is called the inverse,  $B^{-1}$ :

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- The inverse is so special that every entry in the tableau can be calculated using the inverse.

- Define
  - $X_B$  : the set of basic variables
  - $c_B$  : the set of the cost coefficients of the basic variables
  - $c_j$  : cost coefficient of  $x_j$
  - $P_j$  : the column of  $x_j$
  - $b$  : the column of the right hand side of the constraints
- The simplex tableau column associated with variable  $x_j$  can be found as

Basic	$x_j$	Solution
$z$	$C_B B^{-1} P_j - c_j$	$C_B B^{-1} b$
$X_B$	$B^{-1} P_j$	$B^{-1} b$

- For the previous LP model, assume that during the simplex method, the inverse is

$$B^{-1} = \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -1\frac{1}{4} & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix}$$

and the basic variables are  $x_1$ ,  $x_2$ ,  $s_3$ , and  $s_4$ , in this order.

- To construct the corresponding tableau, calculate the reduced costs and columns of the non basic variables:

$$c_{s_1} = c_B B^{-1} P_1 - c_1 = (5, 4, 0, 0) \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -1\frac{1}{4} & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 0 = \frac{3}{4}$$

$$c_{s_2} = c_B B^{-1} P_2 - c_2 = (5, 4, 0, 0) \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -1\frac{1}{4} & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - 0 = \frac{1}{2}$$

$$\text{column of } s_1 = B^{-1}P_1 = \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -1\frac{1}{4} & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ -1/8 \\ 3/8 \\ 1/8 \end{pmatrix}$$

$$\text{column of } s_2 = B^{-1}P_2 = \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -1\frac{1}{4} & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 3/4 \\ -1\frac{1}{4} \\ -3/4 \end{pmatrix}$$

- The solution column is calculated as

$$B^{-1}b = \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -1\frac{1}{4} & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 24 \\ 6 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1\frac{1}{2} \\ 2\frac{1}{2} \\ 1/2 \end{pmatrix}$$

$$c_B B^{-1}b = (5, 4, 0, 0) \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -1\frac{1}{4} & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 24 \\ 6 \\ 1 \\ 2 \end{pmatrix} = 21$$

- The tableau now is

Basic	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	Solution
$z$	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
$x_1$	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
$x_2$	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$1\frac{1}{2}$
$s_3$	0	0	$\frac{3}{8}$	$-1\frac{1}{4}$	1	0	$2\frac{1}{2}$
$s_4$	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

# Treatment of new constraints in the simplex method

- Take the LP model:

$$\begin{aligned}\min z &= 3x_1 + 2x_2 + x_3 \\ \text{s.t.} \quad & 3x_1 + x_2 + x_3 \geq 3 \\ & -3x_1 + 3x_2 + x_3 \geq 6 \\ & x \geq 0\end{aligned}$$

- The LP in the simplex form:

$$\begin{aligned}\min z &= 3x_1 + 2x_2 + x_3 + 100R_1 + 100R_2 \\ \text{s.t.} \quad & 3x_1 + x_2 + x_3 - s_1 + R_1 = 3 \\ & -3x_1 + 3x_2 + x_3 - s_2 + R_2 = 6 \\ & x, s, R \geq 0\end{aligned}$$

- The optimal tableau is

Basic	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$R_1$	$R_2$	Solution
$z$	-3	0	0	-0.5	-0.5	-99.5	-99.5	4.5
$x_3$	6	0	1	-1.5	0.5	1.5	-0.5	1.5
$x_2$	-3	1	0	0.5	-0.5	-0.5	0.5	1.5

- Assume that a new constraint is added to the model:

$$x_1 + x_2 + x_3 \leq 3$$

- First, write the new constraint in the equation form by adding a slack:

$$x_1 + x_2 + x_3 + s_3 = 3$$

- Then, it is added to the optimal tableau in a new row:

Basic	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	Solution
$z$	-3	0	0	-0.5	-0.5	0	4.5
$x_3$	6	0	1	-1.5	0.5	0	1.5
$x_2$	-3	1	0	0.5	-0.5	0	1.5
$s_3$	1	1	1	0	0	1	3

- This tableau is not optimal because the column of  $x_3$  has two elements of 1; same thing for  $x_2$  column.
- Hence, row operations have to be applied to make the tableau optimal:

Basic	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	Solution
$z$	-3	0	0	-0.5	-0.5	0	4.5
$x_3$	6	0	1	-1.5	0.5	0	1.5
$x_2$	-3	1	0	0.5	-0.5	0	1.5
$s_3$	-2	0	0	1	0	1	0

- This tableau is optimal.
- The addition of the new constraint did not change the optimal solution.

## Another example of adding a new constraint

- Take the LP model:

$$\text{Maximize } z = 2x_1 + 3x_2$$

subject to

$$2x_1 + x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

- The LP model in simplex form:

$$\text{Maximize } z = 2x_1 + 3x_2$$

subject to

$$2x_1 + x_2 + s_1 = 4$$

$$x, s \geq 0$$

- The optimal tableau is

Basic	$x_1$	$x_2$	$s_1$	Solution
$z$	4	0	3	12
$x_2$	2	1	1	4

- We want to add the following constraint to the original LP model:

$$x_1 + 2x_2 \leq 5$$

- The LP model now becomes

$$\text{Maximize } z = 2x_1 + 3x_2$$

subject to

$$2x_1 + x_2 + s_1 = 4$$

$$x_1 + 2x_2 + s_2 = 5$$

$$x, s \geq 0$$

- Adding a new constraint is equivalent to adding a new row to the simplex tableau.
- Instead of solving the model from the beginning, we can start from the last optimal tableau and apply the dual simplex if the addition of the new constraint leads to an infeasible solution.

- The last optimal tableau with the addition of the new constraint is

Basic	$x_1$	$x_2$	$s_1$	$s_2$	Solution
$z$	4	0	3	0	12
$x_2$	2	1	1	0	4
$s_2$	1	2	0	1	5

- The tableau is not optimal because the column of the basic variable  $x_2$  has two elements that are different from zero.

- Apply row operations to convert the tableau to optimal:

Basic	$x_1$	$x_2$	$s_1$	$s_2$	Solution
$z$	4	0	3	0	12
$x_2$	2	1	1	0	4
$s_2$	-3	0	-2	1	-3

- This solution is optimal but not feasible.
- Hence, the dual simplex has to be applied.
- First, the leaving basic variable is the one that has a negative value, which is in this tableau  $s_2$ .
- Then, we divide the reduced costs by the absolute values of the negative row coefficients of the leaving basic variable.

- From this tableau, the ratios are

$$\frac{4}{|-3|}, \frac{3}{|-2|}$$

- Therefore, the entering variable is the one that corresponds to the minimum ratio:

$$\min \left\{ \frac{4}{|-3|}, \frac{3}{|-2|} \right\} = \frac{4}{3}$$

- Hence,  $x_1$  becomes basic in the next tableau.

- After applying row operations, the tableau becomes

Basic	$x_1$	$x_2$	$s_1$	$s_2$	Solution
$z$	0	0	$\frac{1}{3}$	$1\frac{1}{3}$	8
$x_2$	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	2
$x_1$	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	1

- The tableau is optimal.