Probabilistic dynamic programming

Objectives of the topic:

- Introduce the probabilistic natures of some problems
- Present the dynamic programming solution when the costs in a stage are uncertain
- See chapter 10 of the text.

The probabilistic nature

- When the parameters of a problem are known with certainty (costs, resource availability, etc), the problem is referred to as deterministic.
- Otherwise, the problem is called probabilistic.
- In real life setting, uncertainties can pop up because of unexpected delays, unexpected failures, or changed mode of the consumer.
- A special case of the probabilistic dynamic programming is when the costs in a stage are uncertain and known only with probabilities.

Probabilistic resource allocation

- The demands from the three production lines are given as

	Demand (ton)	Probability
Line 1	1	0.6
	2	0
	3	0.4
Line 2	1	0.5
	2	0.1
	3	0.4
Line 3	1	0.4
	2	0.3
	3	0.3

- Each ton of production will require one ton of raw material.
- The plant has just purchased 6 tons of raw materials.

- Each ton of production is sold for 2,000 riyals.
- The plant can return the raw materials with an income of 500 riyals per ton.
- The industrial engineer wants to allocate the 6 tons of raw materials to the three production lines so as to maximize the expected net profit.
- Define the following:
  - Stage t is defined as the production lines t, t+1, …, 3.
  - The state x_t at stage t is defined as the tons allocated to production lines t, t+1, …, 3.
  - The alternatives at stage t are the tons assigned to line t, $g_t = 0, 1, 2, 3$.

- Define

  $r_t(g_t)$ = expected revenue earned from g_t tons assigned to line t

- The recursive equation can be defined as

$$f_t(x_t) = \max_{0 \le g_t \le 3} \left\{ r_t(g_t) + f_{t+1}(x_t - g_t) \right\}$$

- Since $g_3 = 6 - g_1 - g_2$ and $g_3 = x_3$, it can be written $f_3(x_3) = r_3(x_3)$.
- Revenues are random because the demands are random.

- The states at stage 3 are 0, 1, 2, and 3.
- The expected revenue at stage 3 can be computed as follows:

$r_3(0) = 0$ (because there is no production)

$r_3(1) = 2$ (because if one ton of raw material is assigned to line 3, it will be consumed to produce one ton of production)

$r_3(2) = 0.4(2+0.5)+0.6(2X2) = 3.4$ (because if two tons of raw materials are assigned to line 3, two cases can happen: (1) the demand is one ton, one ton of materials will be consumed and one ton is sold for 500 riyals with probability of 0.4,

(2) the demand is 2 tons or more, 2 tons of production will be produced with income of 4,000 riyals, with probability of 0.3+0.3=0.6)

$r_3(3) = 0.4(2+0.5+0.5) + 0.3(2X2+0.5) + 0.3(2X3)=4.32$

- In a similar manner, the expected revenues from lines 1 and 2 are:

$r_2(0) = 0$	$r_1(0) = 0$
$r_2(1) = 2$	$r_1(1) = 2$
$r_2(2) = 3.25$	$r_1(2) = 3.10$
$r_2(3) = 4.35$	$r_1(3) = 4.20$

- For stage 2, the possible states are 3, 4, 5, 6.
- The computation of the recursive equation is shown in the table:

	g_2					
x_2	0	1	2	3	$f_2(g_2)$	g^*_2
3	4.34	5.4	5.25	4.35	5.4	1
4	-	6.35	6.65	6.35	6.65	2
5	-	-	7.6	7.75	7.75	3
6	-	-	-	8.7	8.7	3

- Stage 1

	g_1					
x_1	0	1	2	3	$f_1(x_1)$	g^*_1
6	8.7	9.75	9.75	9.6	9.75	1, 2

- Since the last stage has been solved, the solution is terminated.
- From stage 1, there are 2 alternative solutions.
- Arbitrarily choose $g_1 = 1$.
- A total of 5 (=6-1) tons are assigned to lines 2 and 3.
- The maximum return is attained when 3 tons are assigned to line 2, and 2 tons to line 3.