CISE302
Linear Control Systems

Lecture 6:

Blok Diagram and Signal Flow
And State Space Modeling

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(Term 121)

Lecture Objectives

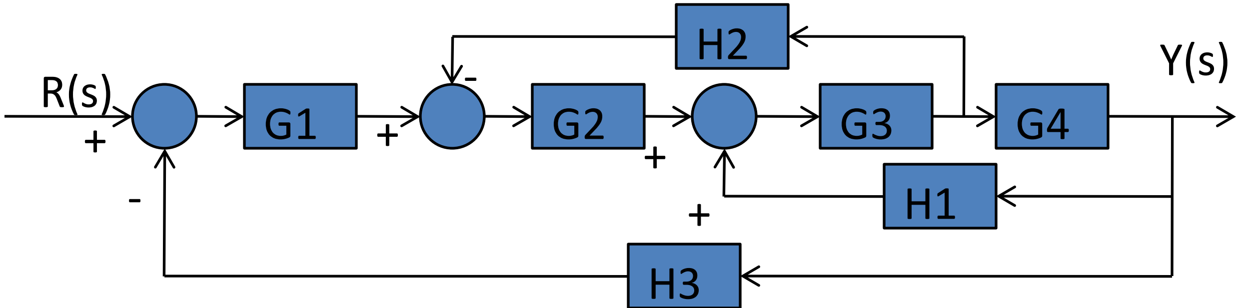
1. To study block diagrams, their components, and their underlying mathematics.
2. To obtain transfer function of systems through block diagram manipulation and reduction.
3. To introduce state space representation from ordinary differential equation model
4. Introduce the relation between state space and transfer function representation.
5. To introduce the signal-flow graphs.
6. To establish a parallel between block diagrams & signal-flow graphs.
7. To use Mason's gain formula for finding transfer function of systems.
8. To introduce state diagrams.
9. To demonstrate the MATLAB tools using case studies.

Applications

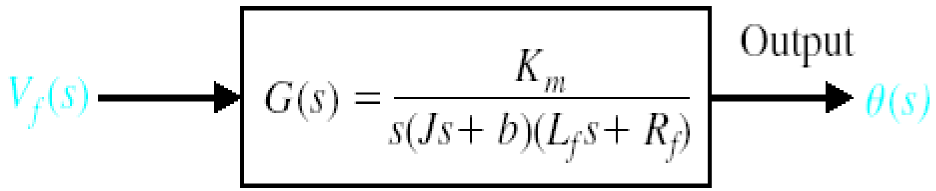
- Spring-mass-damper- Coulomb and viscous damper cases
- RLC circuit, and concept of analogous variables
- Solution of spring-mass-damper (viscous case)
- DC motor- Field current and armature current controlled cases
- Block diagrams of the above DC-motor problems
- Feedback System Transfer
  functions and Signal flow graphs

Block Diagram Reduction

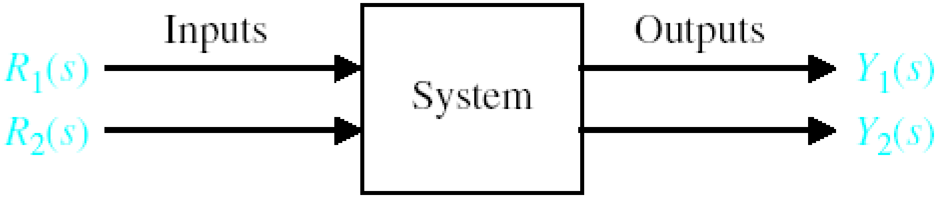
- Combining blocks in a cascade
- Moving a summing point ahead of a block
- Moving summing point behind a block
- Moving splitting point ahead of a block
- Moving splitting point behind a block
- Elimination of a feedback loop



Block Diagram Models

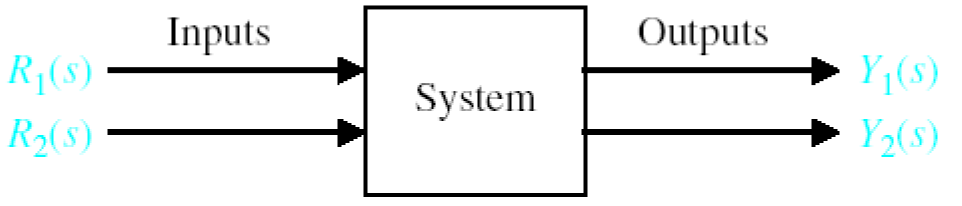


Block diagram of dc motor.

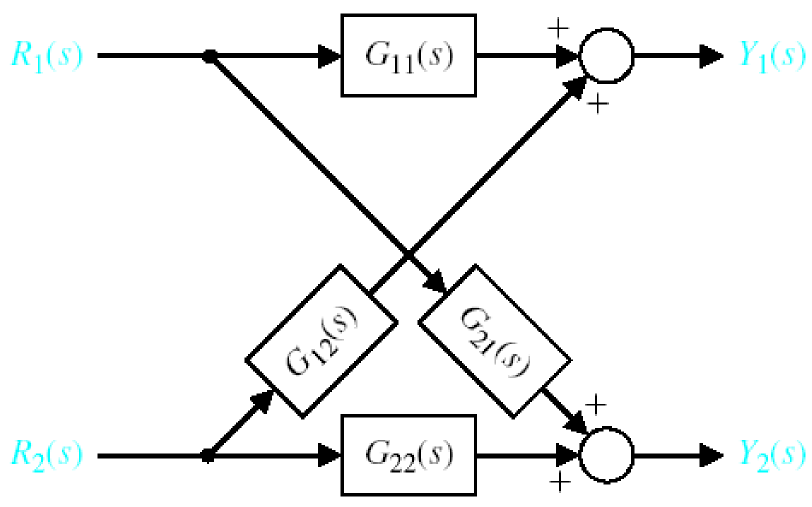


General block representation of two-input, two-output system.

Block Diagram Models



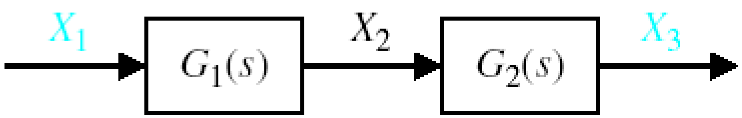
General block representation of two-input, two-output system.



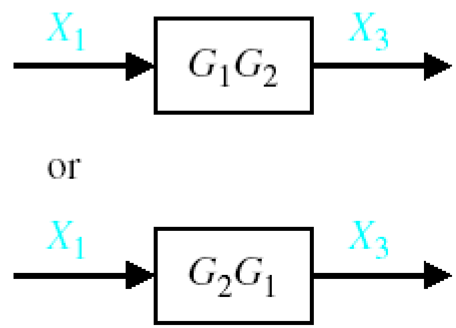
Block diagram of interconnected system.

Block Diagram Models

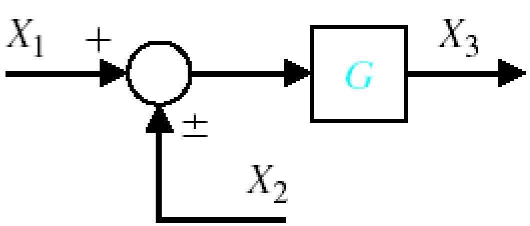
Original Diagram



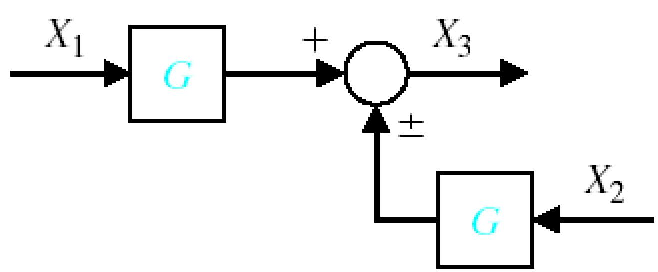
Equivalent Diagram



Original Diagram

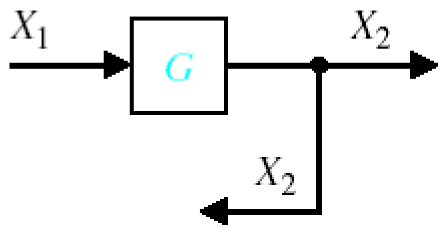


Equivalent Diagram

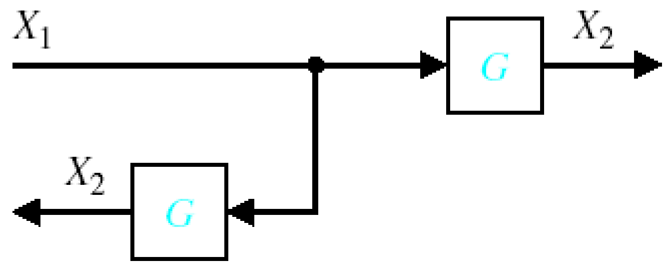


Block Diagram Models

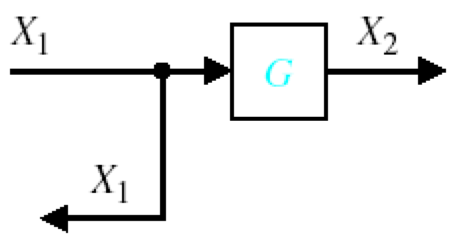
Original Diagram



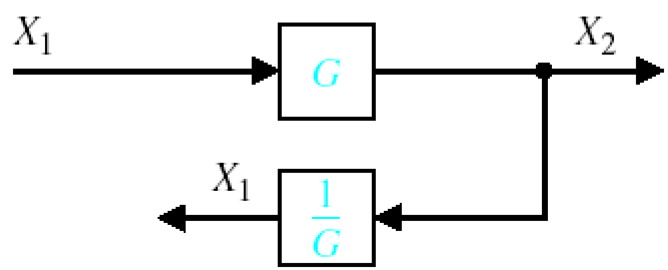
Equivalent Diagram



Original Diagram

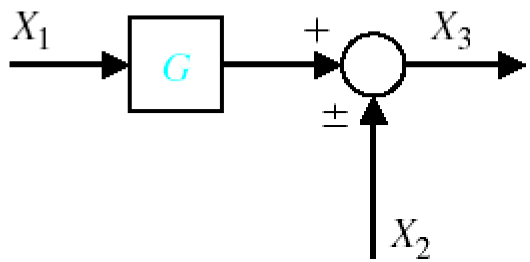


Equivalent Diagram

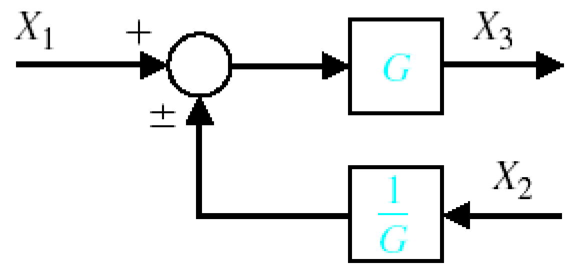


Block Diagram Models

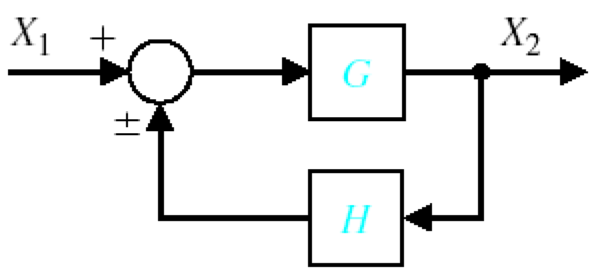
Original Diagram



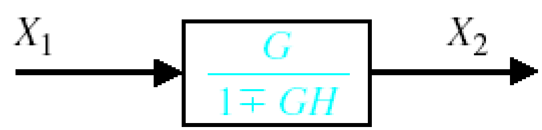
Equivalent Diagram



Original Diagram

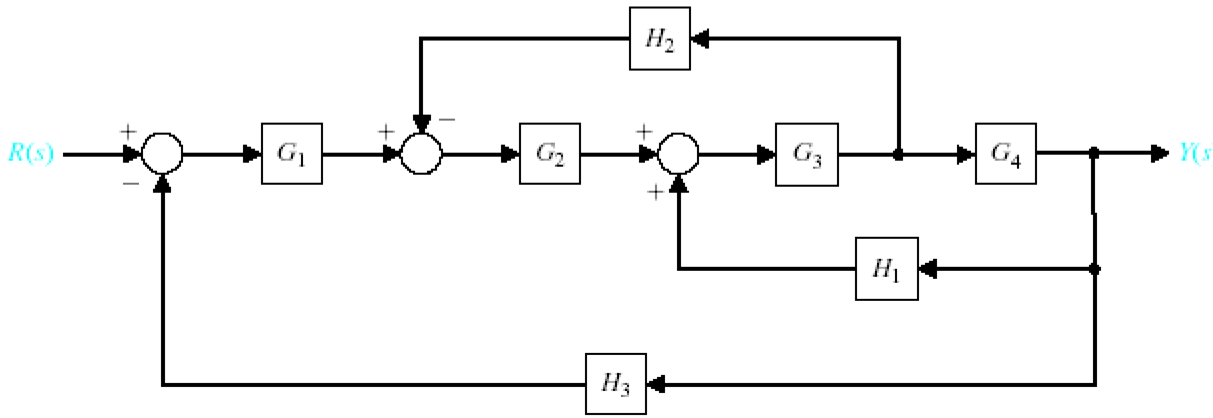


Equivalent Diagram



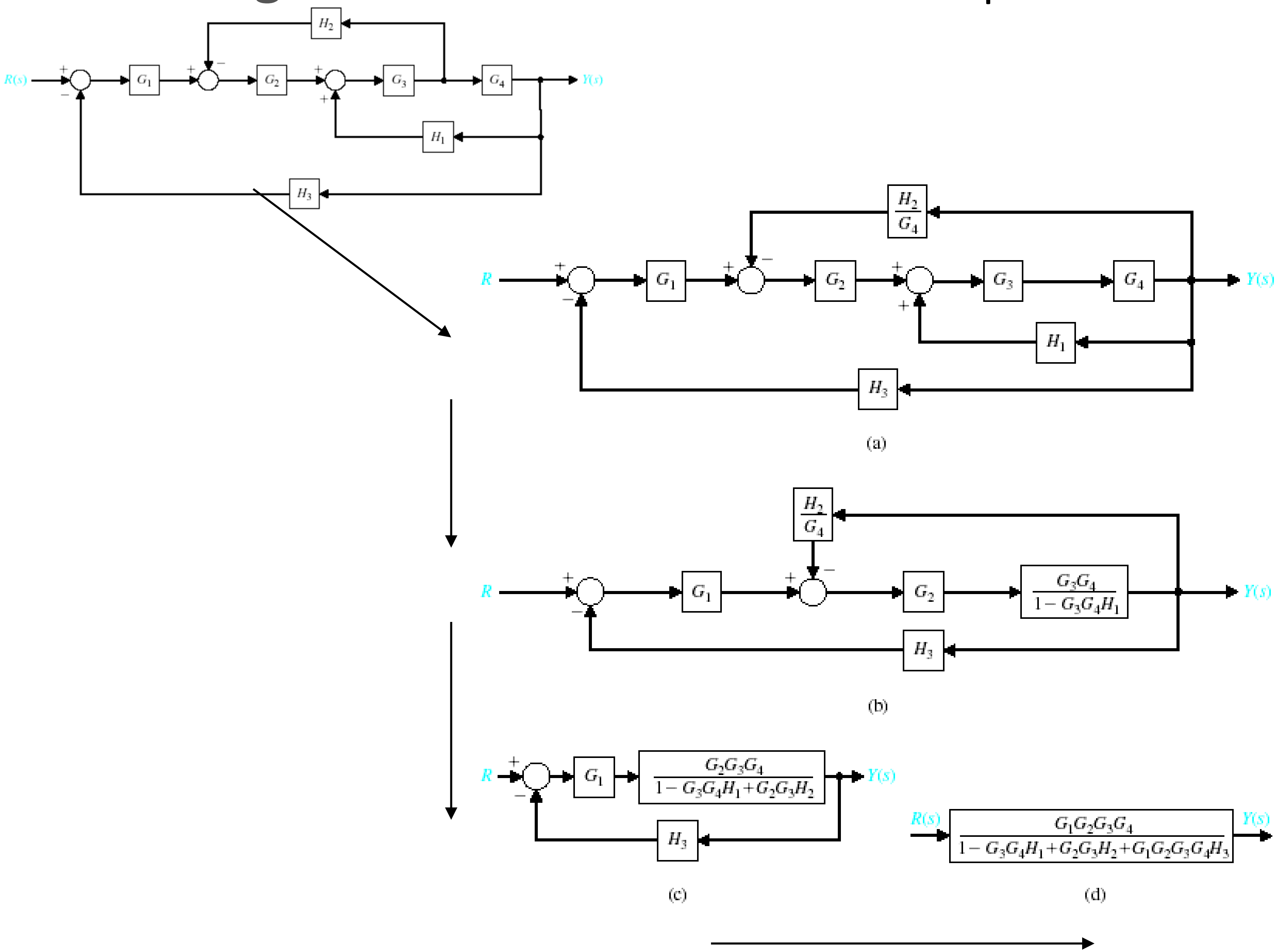
Block Diagram Models

Example 5



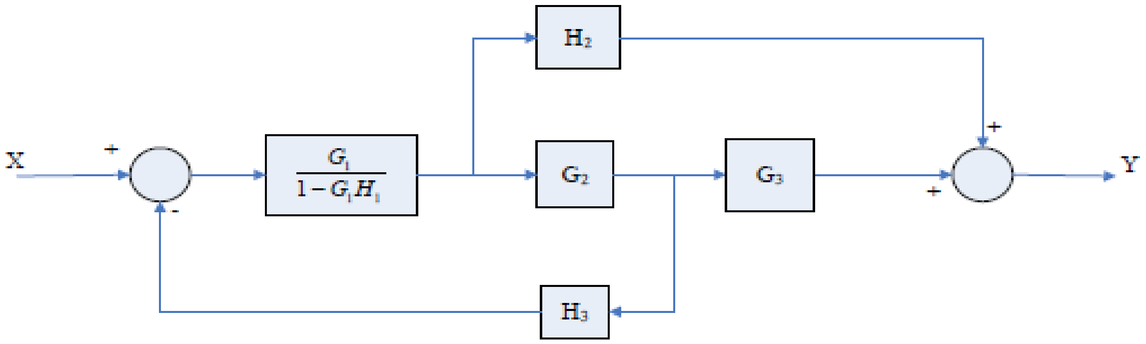
Block Diagram Models

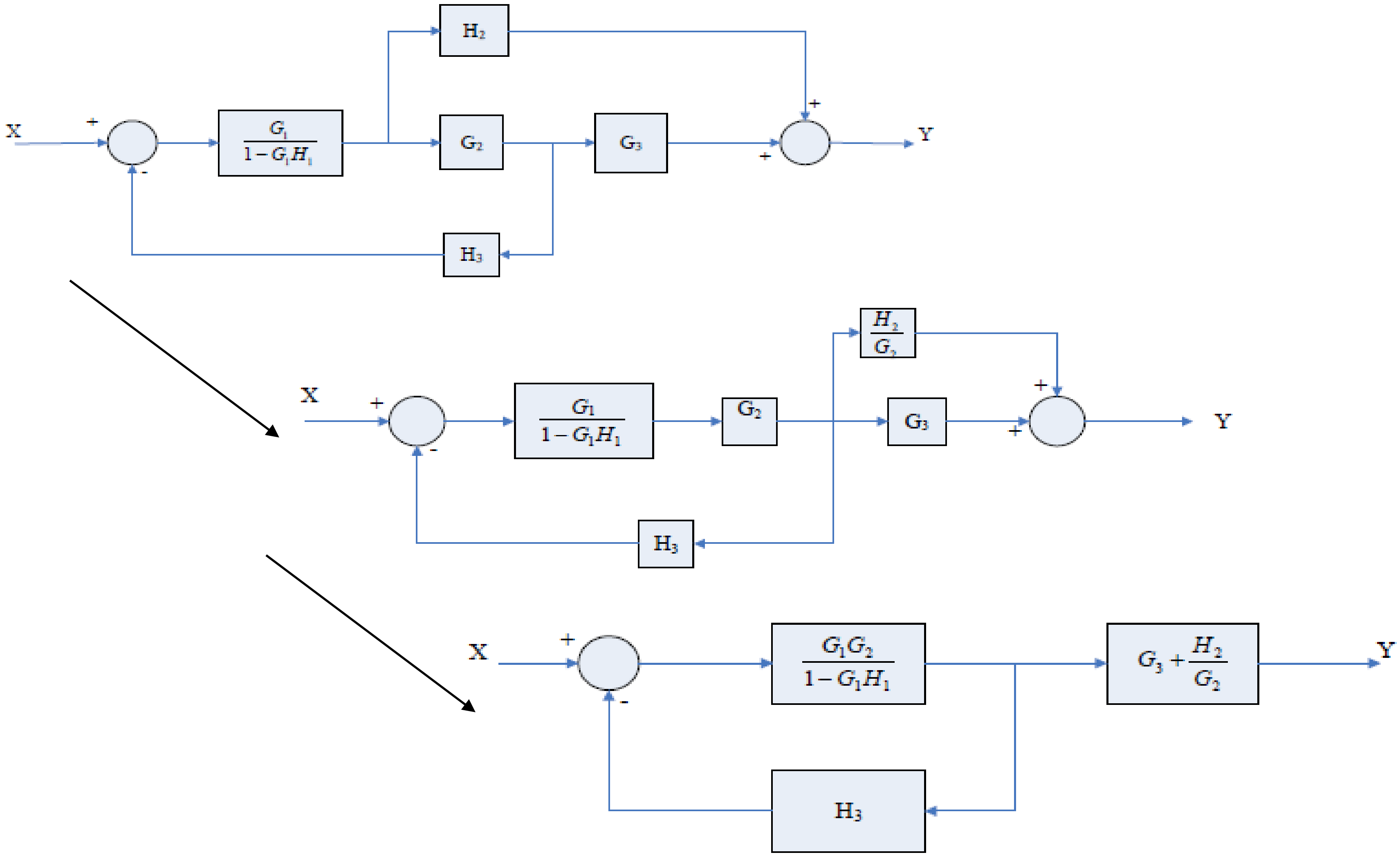
Example 5



Block Diagram Models

Example 6

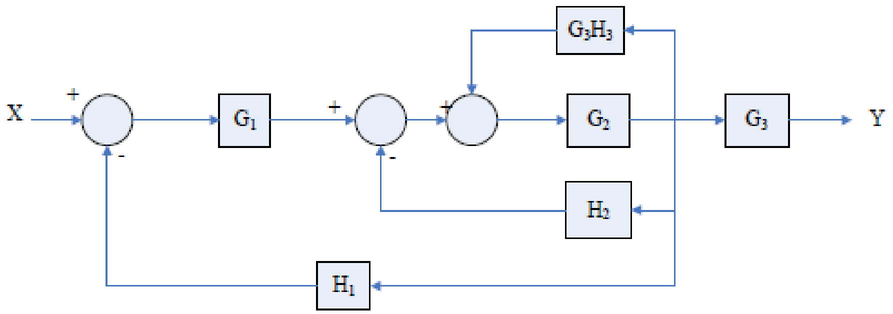


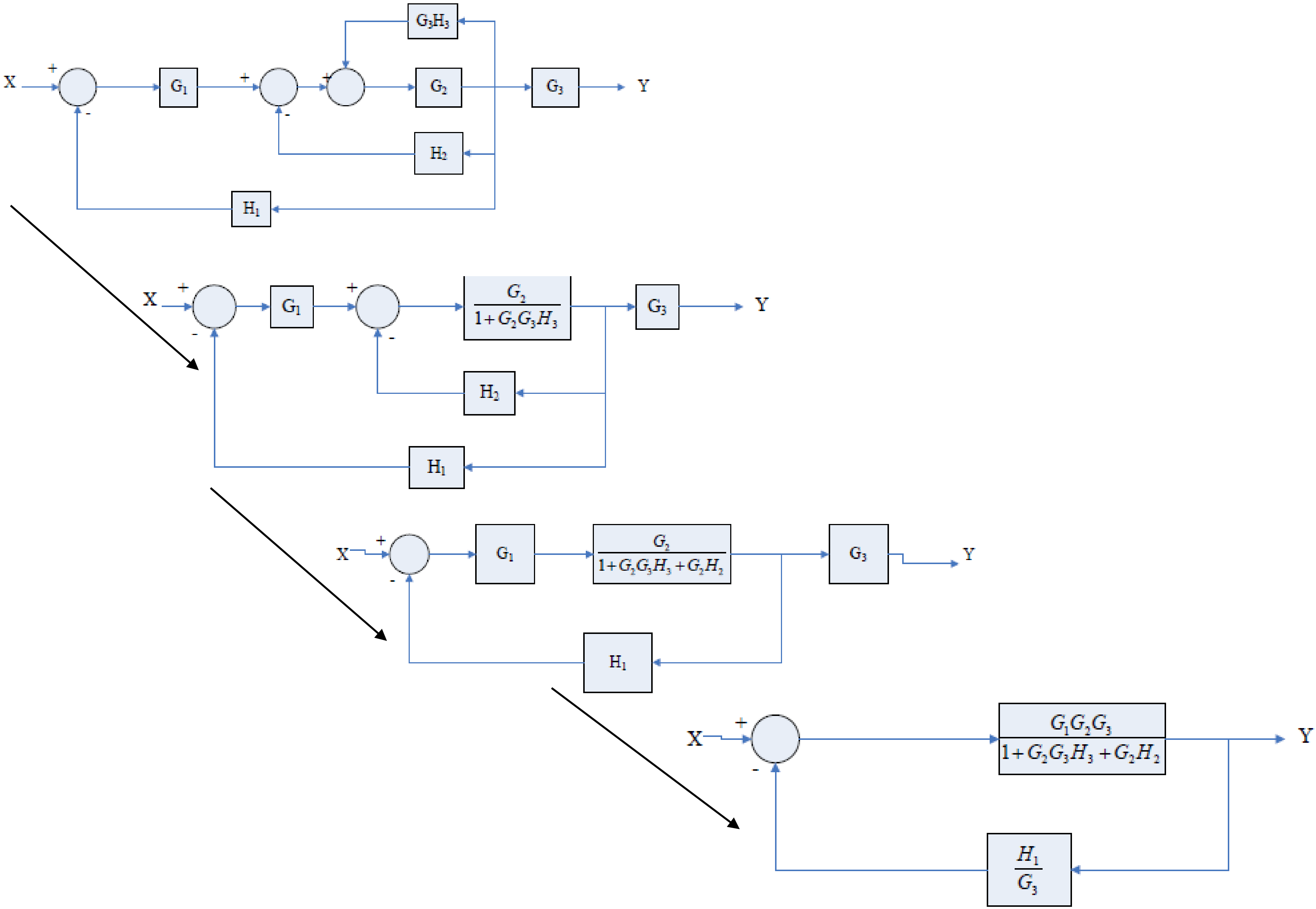


$$\frac{Y(s)}{X(s)} = \frac{\frac{G_1G_2}{1-G_1H_1}}{1+\frac{G_1G_2H_3}{1-G_1H_1}}\left(G_3+\frac{H_2}{G_2}\right) = \frac{G_1G_2G_3+G_1H_2}{1-G_1H_1+G_1G_2H_3}$$

Block Diagram Models

Example 7



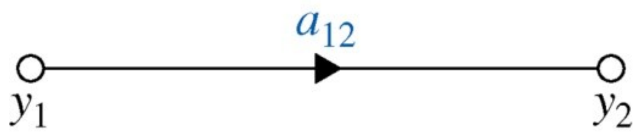


$$\frac{Y(s)}{X(s)} = \frac{G_1G_2G_3}{1 + G_1G_2H_1 + G_2H_2 + G_2G_3H_3}$$

SIGNAL-FLOW GRAPHS (SFGs)

For complex systems, the block diagram method can become difficult to complete. By using the signal-flow graph model, the reduction procedure (used in the block diagram method) is not necessary to determine the relationship between system variables.

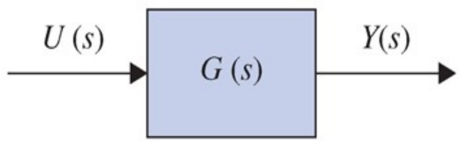
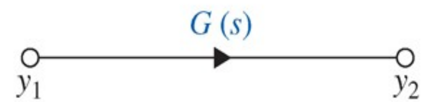
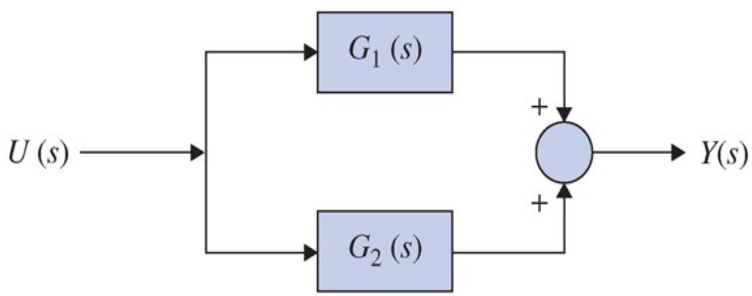
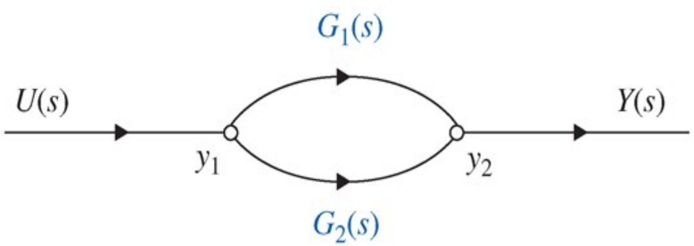
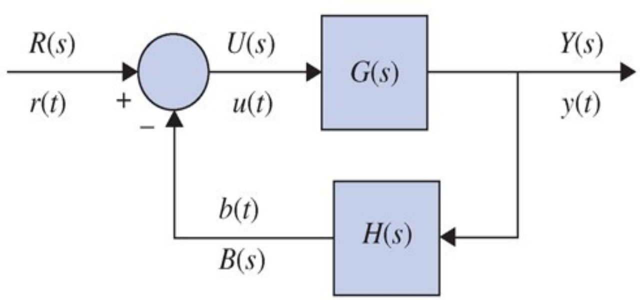
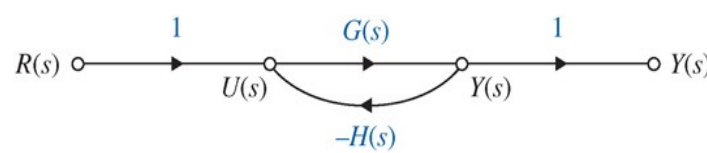
Input Node (Source): *An* input node *is a node that has only outgoing branches*



Output Node (Sink): *An* output node *is a node that has only incoming branches:*

SIGNAL-FLOW GRAPHS (SFGs)

TABLE 3-1 Block diagrams and their SFG equivalent representations

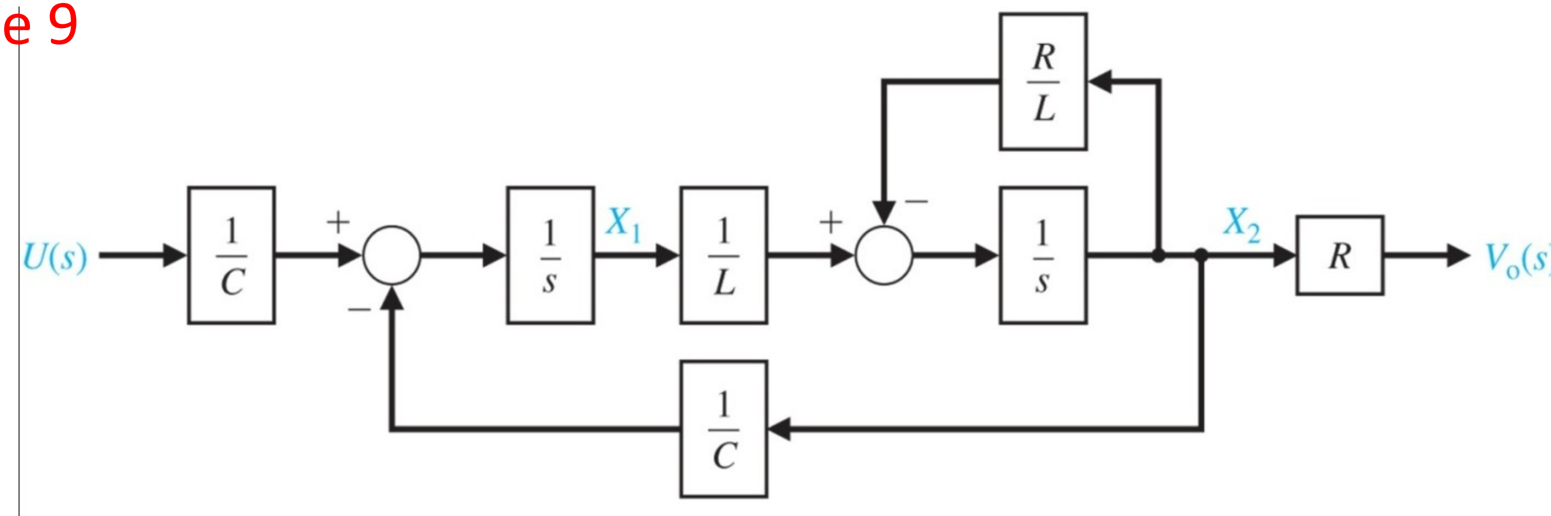
	Block Diagram	Signal Flow Diagram
Simple Transfer Function $\frac{Y(s)}{U(s)} = G(s)$		
Parallel Feedback		
$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$		

SIGNAL-FLOW GRAPHS (SFGs)

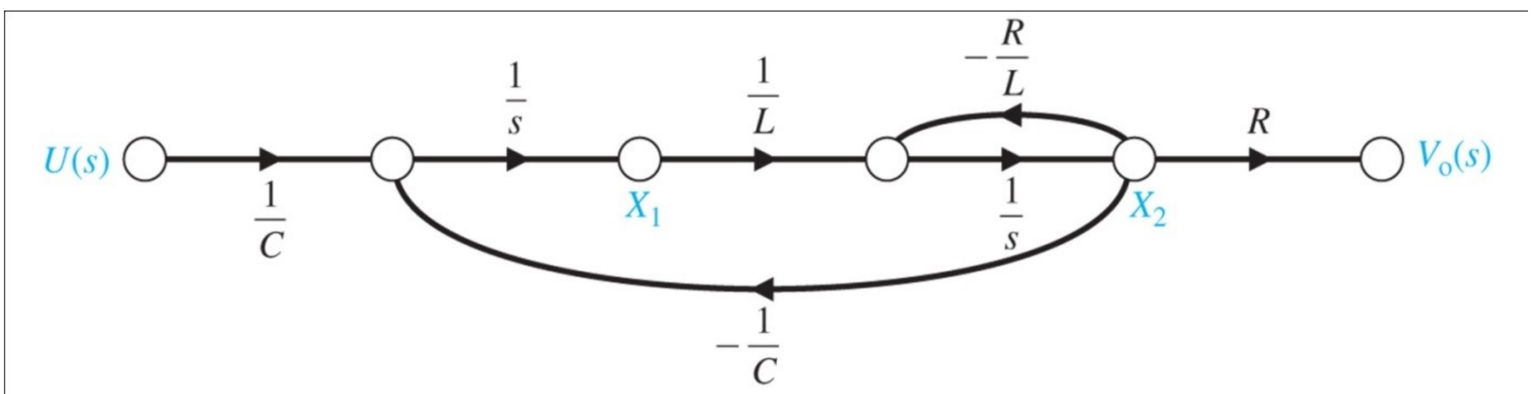
1. SFG applies only to linear systems.
2. The equations for which an SFG is drawn must be algebraic equations in the form of cause-and-effect.
3. Nodes are used to represent variables. Normally, the nodes are arranged from left to right, from the input to the output, following a succession of cause-and-effect relations through the system.
4. Signals travel along branches only in the direction described by the arrows of the branches.
5. The branch directing from node y_k to y_j represents the dependence of y_j upon y_k but not the reverse.
6. A signal y_k traveling along a branch between y_k and y_j is multiplied by the gain of the branch a_{kj}, so a signal $a_{kj}y_k$ is delivered at y_j.

RLC network. (a) Block diagram (b) Signal-flow graph. .

Example 9



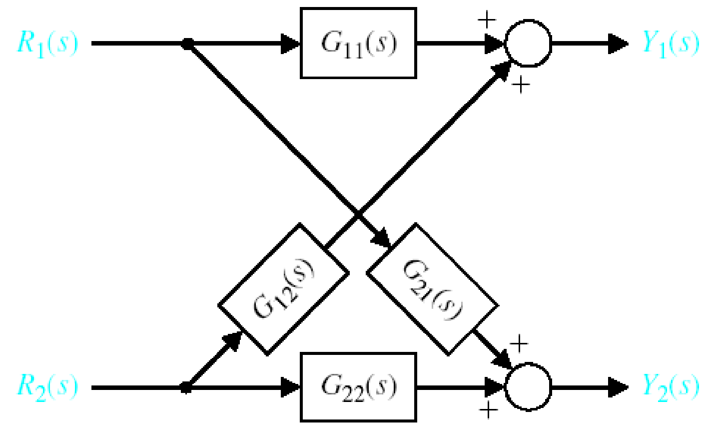
(b)

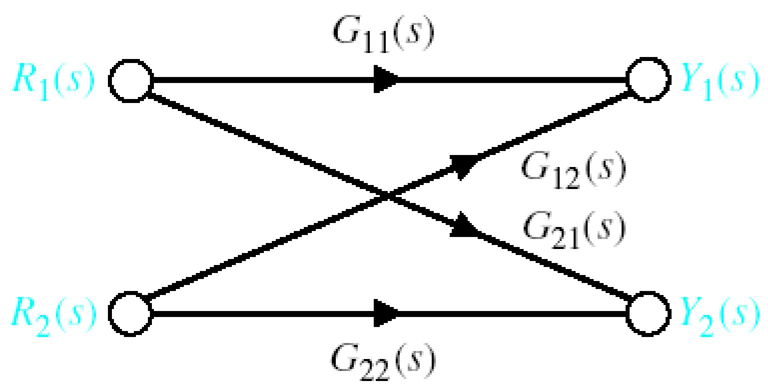


(a)

Signal-Flow Graph Models

Example 10





Signal-flow graph of interconnected system.

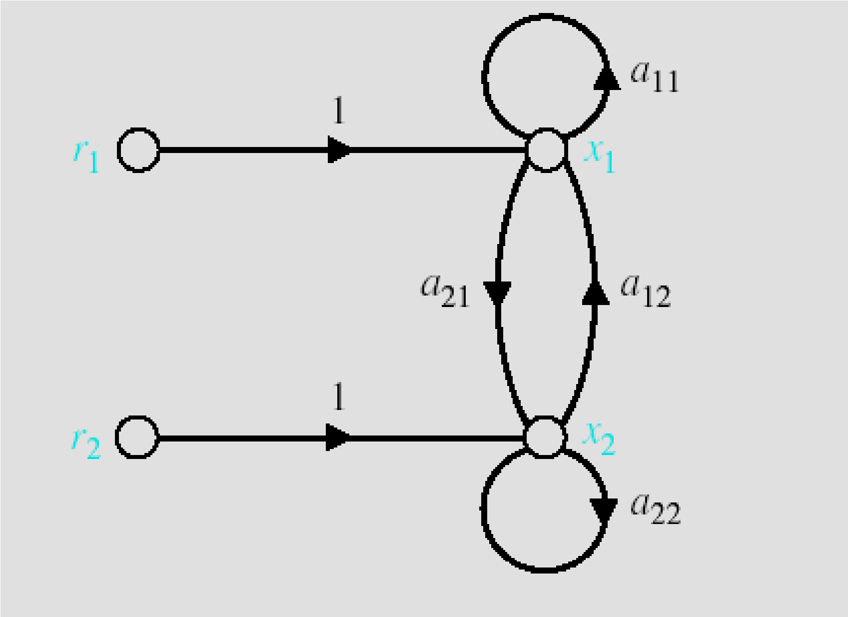
$$Y_1(s) = G_{11}(s)\cdot R_1(s) + G_{12}(s)\cdot R_2(s)$$

$$Y_2(s) = G_{21}(s)\cdot R_1(s) + G_{22}(s)\cdot R_2(s)$$

Signal-Flow Graph Models

Signal-flow graph of two algebraic equations.

Example 11



$$a_{11} \cdot x_1 + a_{12} \cdot x_2 + r_1 = x_1$$

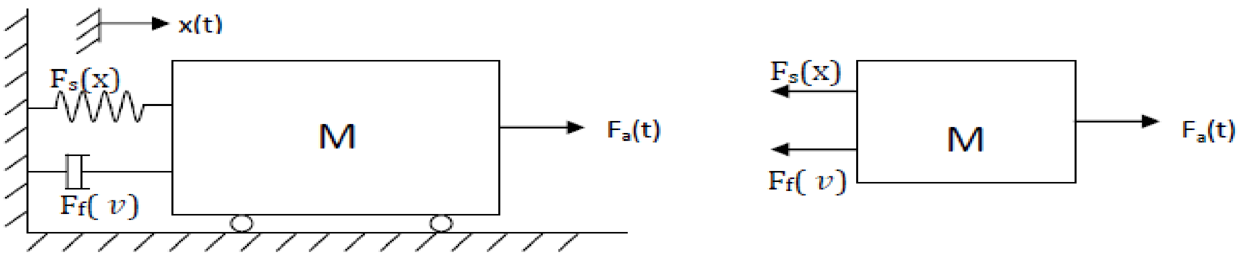
$$a_{21} \cdot x_1 + a_{22} \cdot x_2 + r_2 = x_2$$

State space design

From Lab Manual

Mass-Spring System Model

Consider the following Mass-Spring system shown in the figure. Where $F_s(x)$ is the spring force, $F_f(\dot{x})$ is the friction coefficient, x(t) is the displacement and $F_a(t)$ is the applied force:



Where

$$a = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt^2} \text{ is the acceleration,}$$

$$v = \frac{dx(t)}{dt} \text{ is the speed,}$$

and

$$x(t) \text{ is the displacement.}$$

According to the laws of physics

$$Ma + F_f(v) + F_s(x) = F_a(t) \qquad (1)$$

In the case where:

$$F_f(v) = Bv = B\frac{dx(t)}{dt}$$

$$F_s(x) = Kx(t)$$

The differential equation for the above Mass-Spring system can then be written as follows

$$M\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + Kx(t) = F_a(t) \qquad (2)$$

B is called the friction coefficient and K is called the spring constant.

State space design

Mass-Spring System Example:

Assume the spring force $F_s(x) = Kx^r(t)$. The mass-spring damper is now equivalent to

$$M\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + Kx^r(t) = F_a(t)$$

The second order differential equation has to be decomposed in a set of first order differential equations as follows

Variables	New variable	Differential equation
x(t)	X_1	$\frac{dX_1}{dt} = X_2$
dx(t)/dt	X_2	$\frac{dX_2}{dt} = -\frac{B}{M}X_2 - \frac{K}{M}X_1{}^r(t) + \frac{F_a(t)}{M}$

In vector form, let $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$; $\frac{dX}{dt} = \begin{bmatrix} \frac{dX_1}{dt} \\ \frac{dX_2}{dt} \end{bmatrix}$ then the system can be written as

$$\frac{dX}{dt} = \begin{bmatrix} X_2 \\ -\frac{B}{M}X_2 - \frac{K}{M}X_1{}^r(t) + \frac{F_a(t)}{M} \end{bmatrix}$$

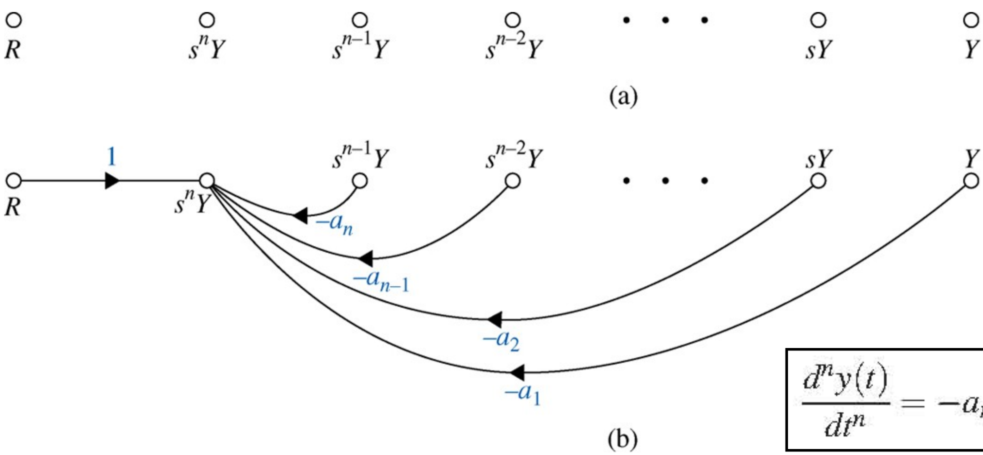
What are X1 and X2 ?

Sections 3.3 to 3.6 from Dorf Text

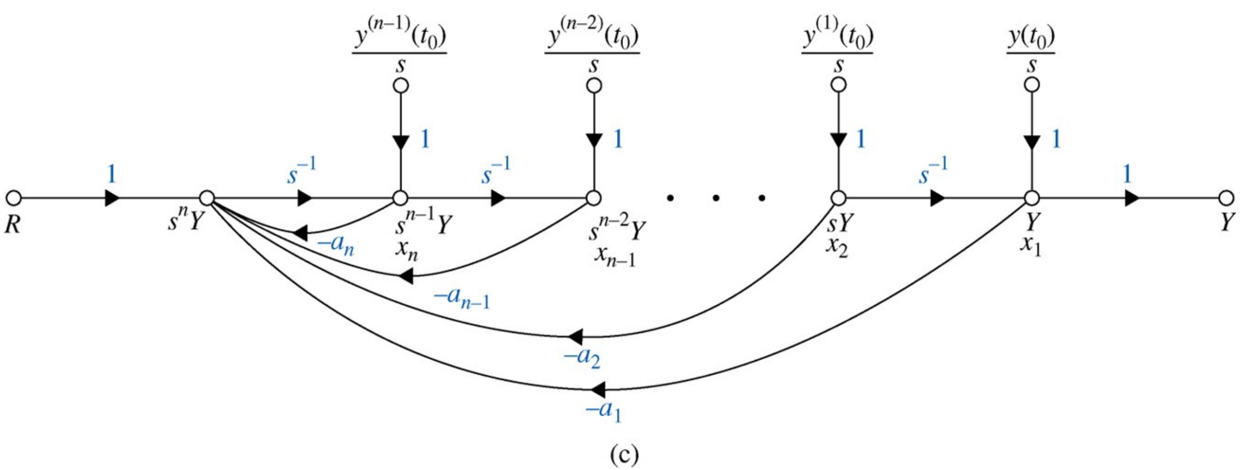
State space design

From Differential Equations to State Diagrams

$$\frac{d^n y(t)}{dt^n} + a_n \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_2 \frac{dy(t)}{dt} + a_1 y(t) = r(t) \qquad (10\text{-}24)$$



$$\frac{d^n y(t)}{dt^n} = -a_n \frac{d^{n-1} y(t)}{dt^{n-1}} - \cdots - a_2 \frac{dy(t)}{dt} - a_1 y(t) + r(t) \qquad (10\text{-}25)$$



State-diagram representation of the differential equation

State space design

State equation

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + \cdots + b_{1m}u_m,$$
$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + \cdots + b_{2m}u_m,$$
$$\vdots$$
$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + \cdots + b_{nm}u_m,$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

the state variables $(x_1, x_2, \ldots, x_n)$ and the inputs $(u_1, u_2, \ldots, u_m)$.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12}\cdots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2n} \\ \vdots & \cdots & \vdots \\ a_{n1} & a_{n2}\cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11}\cdots b_{1m} \\ \vdots \quad\quad \vdots \\ b_{n1}\cdots b_{nm} \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}.$$

$$\boxed{\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}.}$$

output equation

$$\boxed{\mathbf{y} = \mathbf{Cx} + \mathbf{Du},}$$

State space design

First order scalar ODE

Example 12

Consider the first-order differential equation

$$\dot{x} = ax + bu, \tag{3.20}$$

Taking the Laplace transform of Equation (3.20), we have

$$sX(s) - x(0) = aX(s) + bU(s);$$

therefore,

$$X(s) = \frac{x(0)}{s-a} + \frac{b}{s-a}U(s). \tag{3.21}$$

The inverse Laplace transform of Equation (3.21) can be shown to be

$$x(t) = e^{at}x(0) + \int_0^t e^{+a(t-\tau)}bu(\tau)\,d\tau. \tag{3.22}$$

State space design

First order n-dimension system of ODEs

$$e^{\mathbf{A}t} = \exp(\mathbf{A}t) = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots + \frac{\mathbf{A}^kt^k}{k!} + \cdots, \tag{3.23}$$

which converges for all finite t and any $\mathbf{A}$ [2]. Then the solution of the state differential equation is found to be

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0) + \int_0^t \exp[\mathbf{A}(t-\tau)]\mathbf{B}\mathbf{u}(\tau)\,d\tau. \tag{3.24}$$

Equation (3.24) may be verified by taking the Laplace transform of Equation (3.16) and rearranging to obtain

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) + [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s), \tag{3.25}$$

term on the right-hand side involves the product $\Phi(s)\mathbf{B}\mathbf{U}(s)$, we obtain Equation (3.24). The matrix exponential function describes the unforced response of the system and is called the **fundamental** or **state transition matrix** $\Phi(t)$. Thus, Equation (3.24) can be written as

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) + \int_0^t \Phi(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau. \tag{3.26}$$

The solution to the unforced system (that is, when $\mathbf{u} = 0$) is simply

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} \phi_{11}(t) & \cdots & \phi_{1n}(t) \\ \phi_{21}(t) & \cdots & \phi_{2n}(t) \\ \vdots & & \vdots \\ \phi_{n1}(t) & \cdots & \phi_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix}. \tag{3.27}$$

State space design

CONVERTING FROM STATE SPACE TO A TRANSFER FUNCTION

Given the state and output equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{3.68a}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{3.68b}$$

take the Laplace transform assuming zero initial conditions:[8]

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \tag{3.69a}$$

$$\mathbf{Y}(\mathbf{s}) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(\mathbf{s}) \tag{3.69b}$$

Solving for $\mathbf{X}(s)$ in Eq. (3.69a),

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s) \tag{3.70}$$

or

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \tag{3.71}$$

State space design

$$s\mathbf{X}(s) = \mathbf{AX}(s) + \mathbf{BU}(s) \tag{3.69a}$$

$$\mathbf{Y}(\mathbf{s}) = \mathbf{CX}(s) + \mathbf{DU}(\mathbf{s}) \tag{3.69b}$$

Solving for $\mathbf{X}(s)$ in Eq. (3.69a),

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{BU}(s) \tag{3.70}$$

or

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{BU}(s) \tag{3.71}$$

where $\mathbf{I}$ is the identity matrix.

Substituting Eq. (3.71) into Eq. (3.69*b*) yields

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{BU}(s) + \mathbf{DU}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s) \tag{3.72}$$

We call the matrix $[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]$ the transfer function matrix, since it relates the output vector, $\mathbf{Y}(s)$, to the input vector, $\mathbf{U}(s)$. However, if $\mathbf{U}(s) = U(s)$ and $\mathbf{Y}(s) = Y(s)$ are scalars, we can find the transfer function,

$$\boxed{T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}} \tag{3.73}$$

State space design

CHARACTERISTIC EQUATIONS, EIGENVALUES, AND EIGENVECTORS

Characteristic Equation from a Differential Equation

Consider that a linear time-invariant system is described by the differential equation

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_m\frac{d^m u(t)}{dt^m} + b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}} + \cdots + b_1\frac{du(t)}{dt} + b_0 u(t) \quad (10\text{-}143)$$

where $n > m$. By defining the operator s as

$$s^k = \frac{d^k}{dt^k} \qquad k = 1, 2, \ldots, n \quad (10\text{-}144)$$

Eq. (10-143) is written

$$\left(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0\right)y(t) = \left(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0\right)u(t) \quad (10\text{-}145)$$

The **characteristic equation** of the system is defined as

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \quad (10\text{-}146)$$

which is obtained by setting the homogeneous part of Eq. (10-145) to zero.

How to solve High Order ODE

- How do solve second order ODE?

$$\ddot{x} + 3\dot{x} + 6x = 1$$

- How do solve high order ODE?

The general approach to solve ODEs

Example 13

high order ODE → convert → System of first order ODE → solve

$$\ddot{x} + 3\dot{x} + 6x = 1$$

$$\dot{x}(0) = 1; x(0) = 4$$

Second order ODE

convert →

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ 1 - 3z_2 - 6z_1 \end{bmatrix},$$

$$Z(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Two first order ODEs

solve →

State space design

Example 14

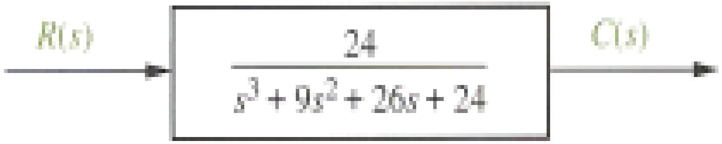
Converting a transfer function with constant term in numerator

Problem: Find the state-space representation in phase-variable form for the transfer function shown in Figure 3.10(*a*).

SOLUTION:

Step 1 Find the associated differential equation. Since

$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)} \tag{3.54}$$



(*a*)

State space design

Example 14

cross-multiplying yields

$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s) \tag{3.55}$$

The corresponding differential equation is found by taking the inverse Laplace transform, assuming zero initial conditions:

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r \tag{3.56}$$

Step 2 Select the state variables.
Choosing the state variables as successive derivatives, we get

$$x_1 = c \tag{3.57a}$$

$$x_2 = \dot{c} \tag{3.57b}$$

$$x_3 = \ddot{c} \tag{3.57c}$$

Differentiating both sides and making use of Eqs. (3.57) to find $\dot{x}_1$ and $\dot{x}_2$, and Eq. (3.56) to find $\dddot{c} = \dot{x}_3$, we obtain the state equations. Since the output is $c = x_1$, the combined state and output equations are

$$\dot{x}_1 = x_2 \tag{3.58a}$$

$$\dot{x}_2 = x_3 \tag{3.58b}$$

$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r \tag{3.58c}$$

$$y = c = x_1 \tag{3.58d}$$

EXAMPLE

State space design

In vector-matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r \tag{3.59a}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{3.59b}$$

Notice that the third row of the system matrix has the same coefficients as the denominator of the transfer function but negative and in reverse order.

Example 15

State space design

State-space representation to transfer function

Problem: Given the system defined by Eqs. (3.74), find the transfer function, $T(s) = Y(s)/U(s)$, where $U(s)$ is the input and $Y(s)$ is the output.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u \tag{3.74a}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x} \tag{3.74b}$$

SOLUTION: The solution revolves around finding the term $(s\mathbf{I} - \mathbf{A})^{-1}$ in Eq. (3.73).[9] All other terms are already defined. Hence, first find $(s\mathbf{I} - \mathbf{A})$:

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix} \tag{3.75}$$

Now form $(s\mathbf{I} - \mathbf{A})^{-1}$:

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\operatorname{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})} = \frac{\begin{bmatrix} (s^2 + 3s + 2) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3 + 3s^2 + 2s + 1} \tag{3.76}$$

Substituting $(s\mathbf{I} - \mathbf{A})^{-1}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ into Eq. (3.73), where

$$\mathbf{B} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{D} = 0$$

we obtain the final result for the transfer function:

$$T(s) = \frac{10(s^2 + 3s + 2)}{s^3 + 3s^2 + 2s + 1} \tag{3.77}$$

State space design

Example 16

Consider the differential equation

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = r(t) \tag{10-26}$$

Equating the highest-ordered term of the last equation to the rest of the terms, we have

$$\frac{d^2y(t)}{dt^2} = -3\frac{dy(t)}{dt} - 2y(t) + r(t) \tag{10-27}$$

Following the procedure just outlined, the state diagram of the system is drawn as shown in Fig. 10-6. The state variables x_1 and x_2 are assigned as shown.

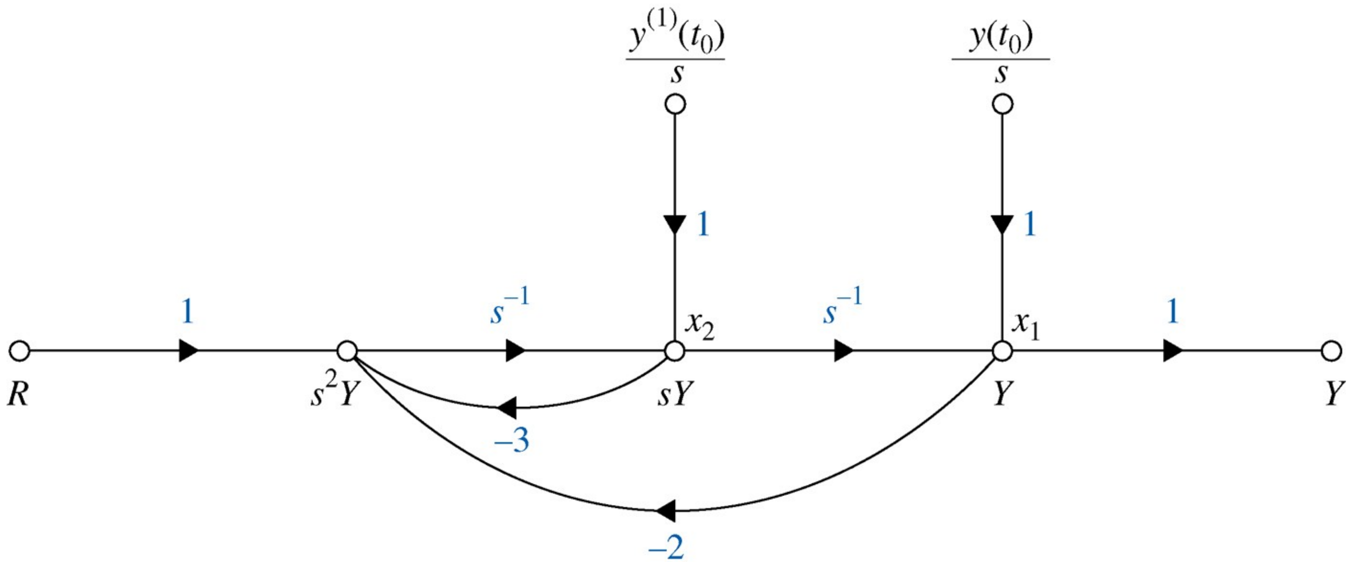


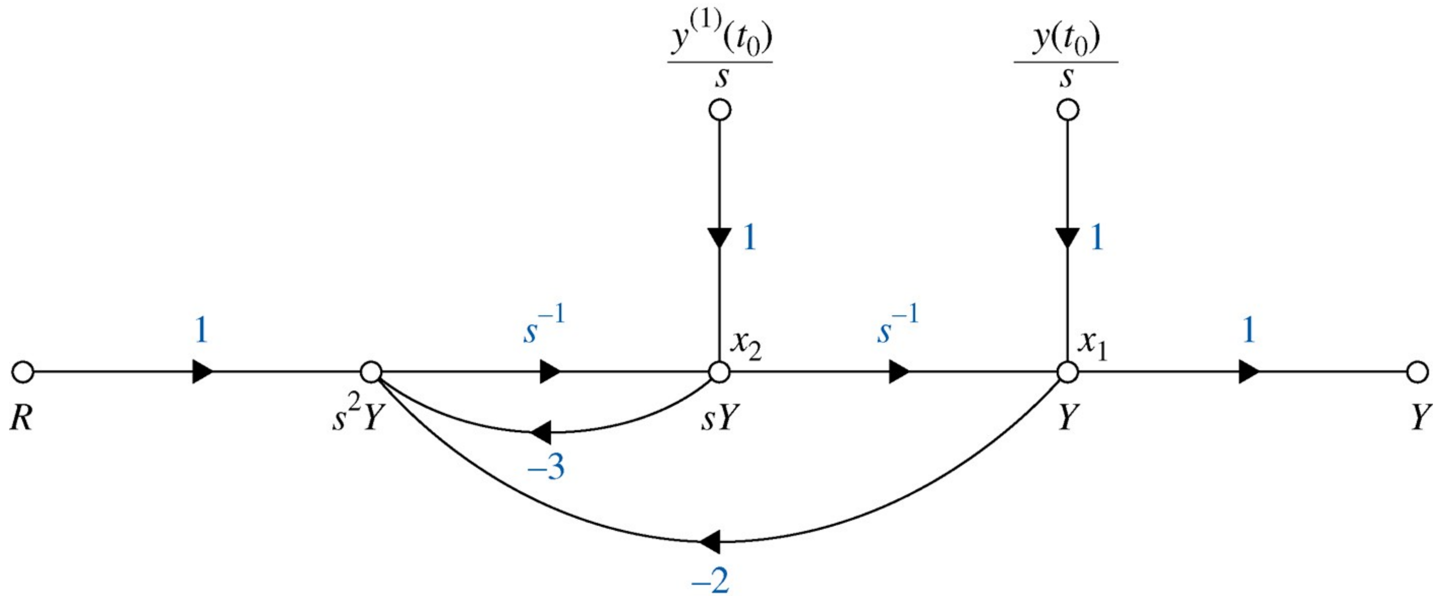
Figure 10-6 State diagram for Eq. (10-26).

State space design

From State Diagrams to Transfer Functions

EXAMPLE

Consider the state diagram of Fig. 10-6. The transfer function between $R(s)$ and $Y(s)$ is obtained by applying the gain formula between these two nodes and setting the initial states to zero. We have



State diagram

Transfer Function: $$\frac{Y(s)}{R(s)} = \frac{1}{s^2 + 3s + 2}$$

State space design

Example 17

Fig. 10-7 shows the state diagram of Fig. 10-6 with the integrator branches and the initial states eliminated. Using $dx_1(t)/dt$ and $dx_2(t)/dt$ as the output nodes and $x_1(t)$, $x_2(t)$, and $r(t)$ as input nodes, and applying the gain formula between these nodes, the state equations are obtained as

$$\frac{dx_1(t)}{dt} = x_2(t) \tag{10-31}$$

$$\frac{dx_2(t)}{dt} = -2x_1(t) - 3x_2(t) + r(t) \tag{10-32}$$

Applying the gain formula with $x_1(t)$, $x_2(t)$, and $r(t)$ as input nodes and $y(t)$ as the output node, the output equation is written

$$y(t) = x_1(t) \tag{10-33}$$

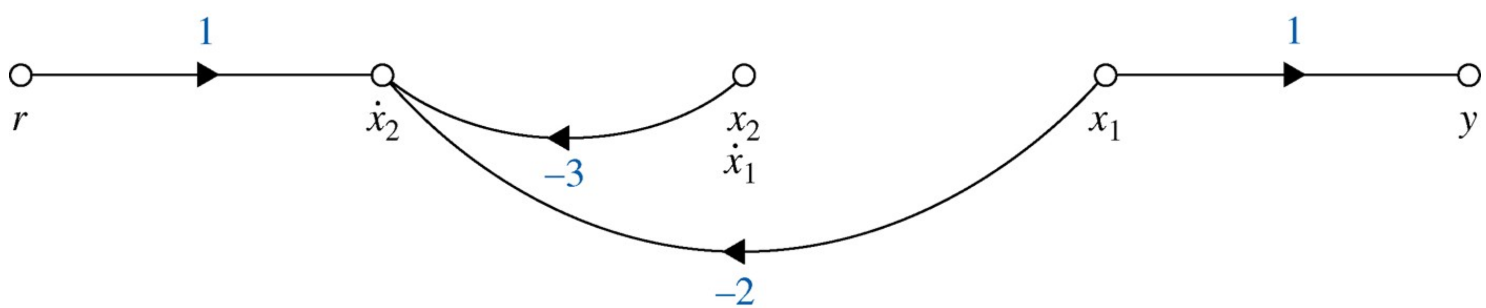


Figure 10-7 State diagram of Fig. 10-6 with the initial states and the integrator branches left out.

From State Diagrams to State and Output Equations

State equation:

$$\frac{dx(t)}{dt} = ax(t) + br(t) \tag{10-29}$$

Output equation:

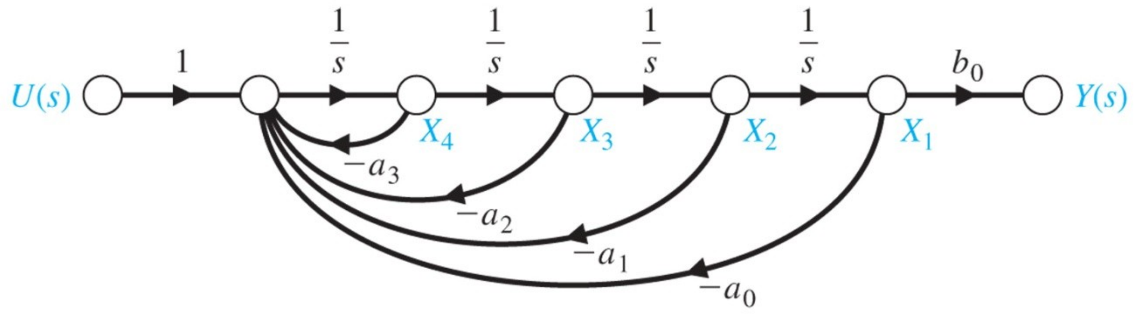
$$y(t) = cx(t) + dr(t) \tag{10-30}$$

where $x(t)$ is the state variable; $r(t)$ is the input; $y(t)$ is the output; and a, b, c, and d are constant coefficients. Based on the general form of the state and output equations, the following procedure of deriving the state and output equations from the state diagram are outlined:

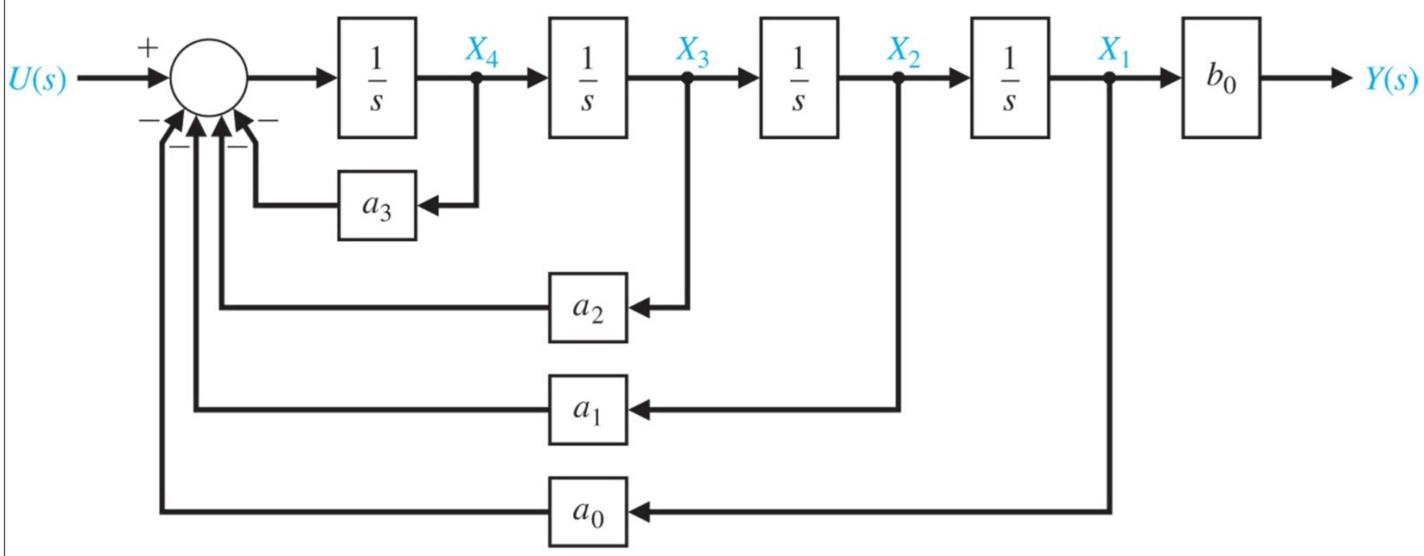
1. Delete the initial states and the integrator branches with gains s^{-1} from the state diagram, since the state and output equations do not contain the Laplace operator s or the initial states.
2. For the state equations, regard the nodes that represent the derivatives of the state variables as output nodes, since these variables appear on the left-hand side of the state equations. The output $y(t)$ in the output equation is naturally an output node variable.
3. Regard the state variables and the inputs as input variables on the state diagram, since these variables are found on the right-hand side of the state and output equations.
4. Apply the SFG gain formula to the state diagram.

Model for $G(s)$ of Equation (3.45). (a) Signal-flow graph. (b) Block diagram.

Example 18



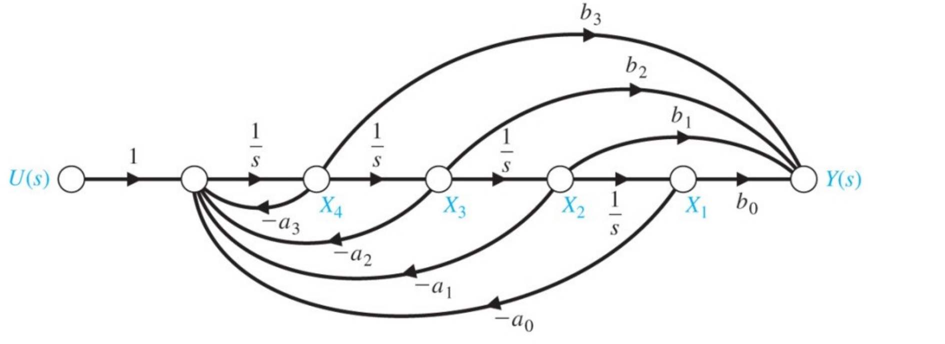
(a)



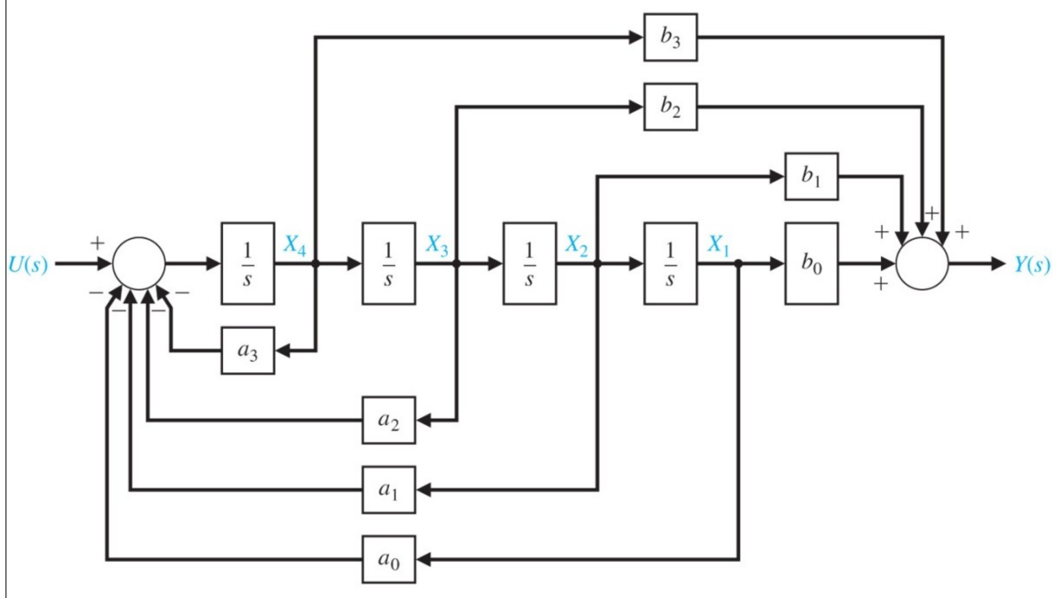
(b)

Model for $G(s)$ of Equation (3.46) in the phase variable format. (a) Signal-flow graph. (b) Block diagram.

Example 19



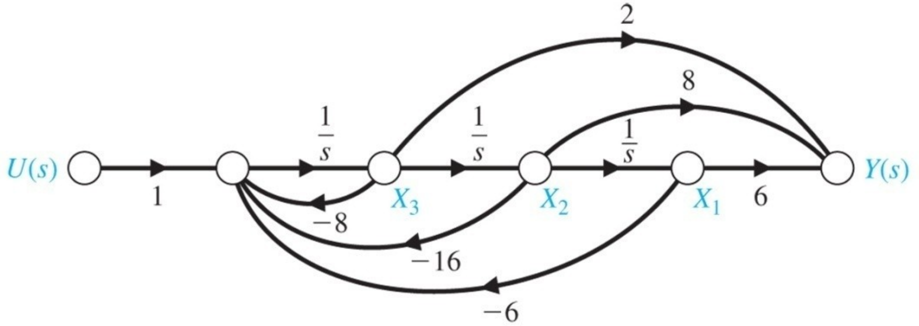
(a)



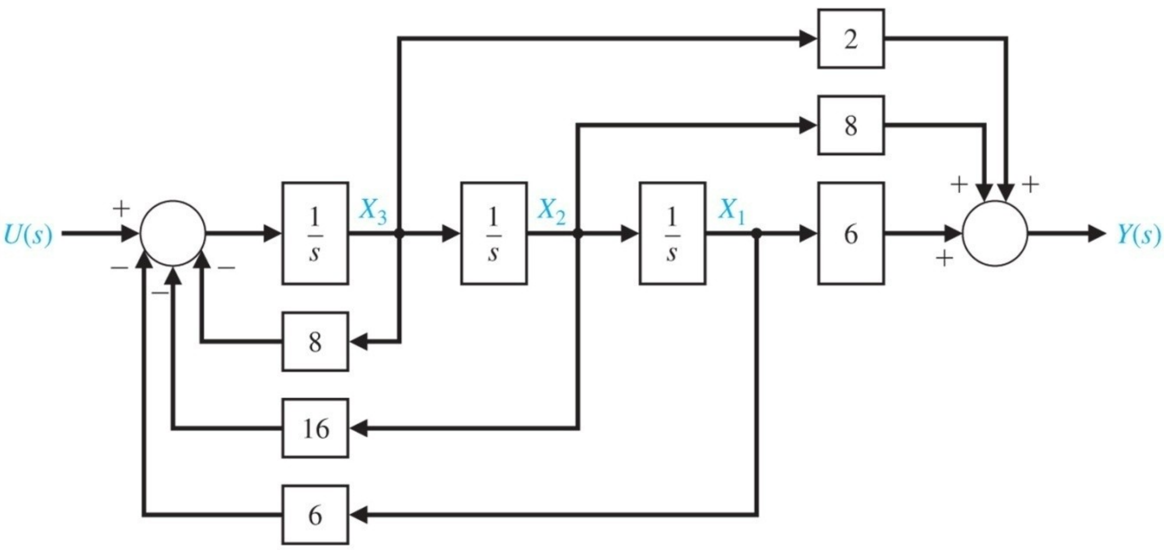
(b)

(a) Phase variable flow graph state model for $T(s)$. (b) Block diagram for the phase variable canonical form.

Example 21



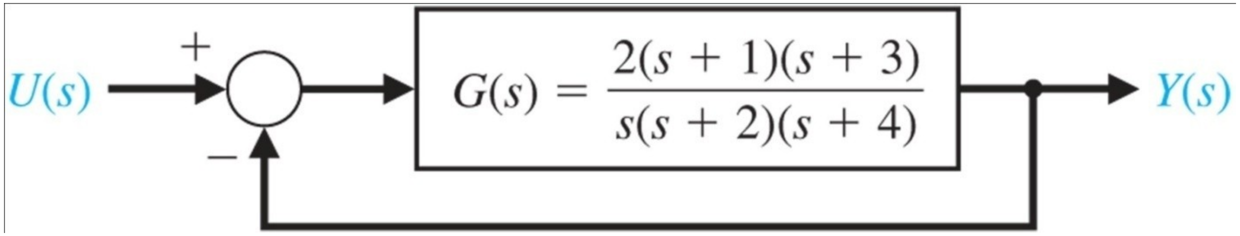
(a)



(b)

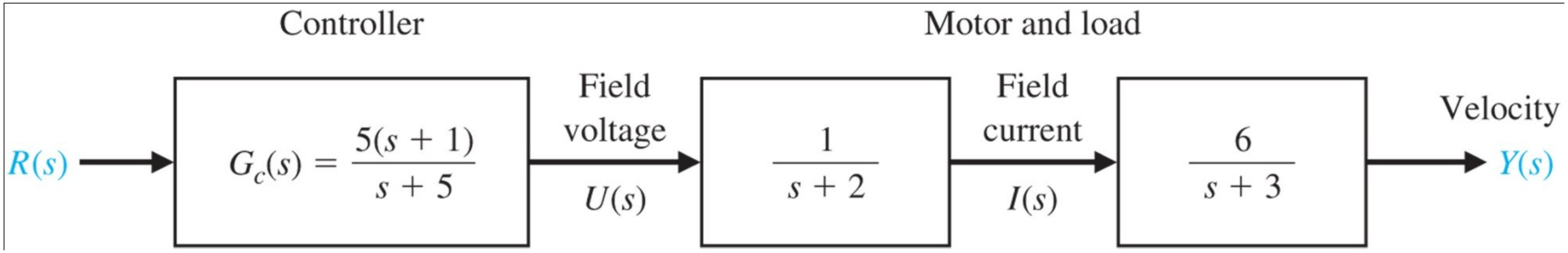
Single-loop control system.

Example 22



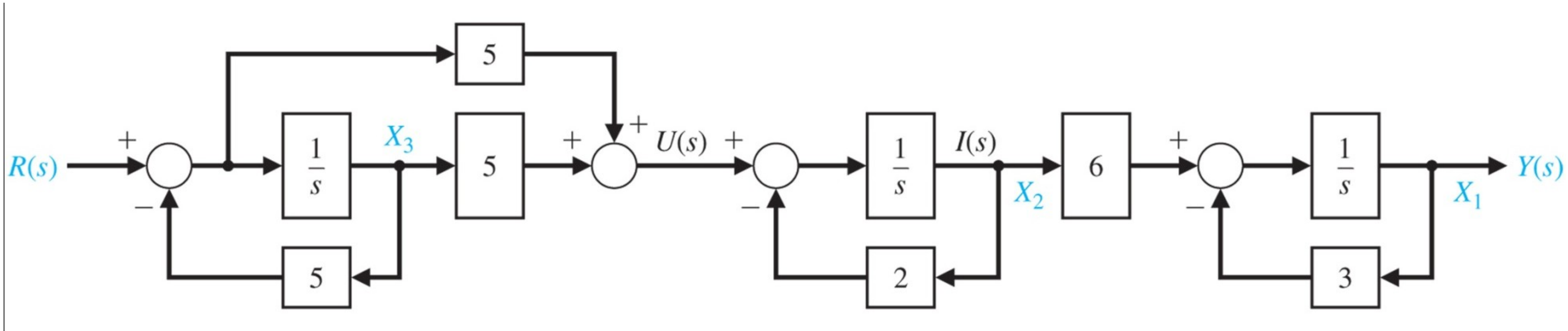
A block diagram model of an open-loop DC motor control with velocity as the output.

Example 23

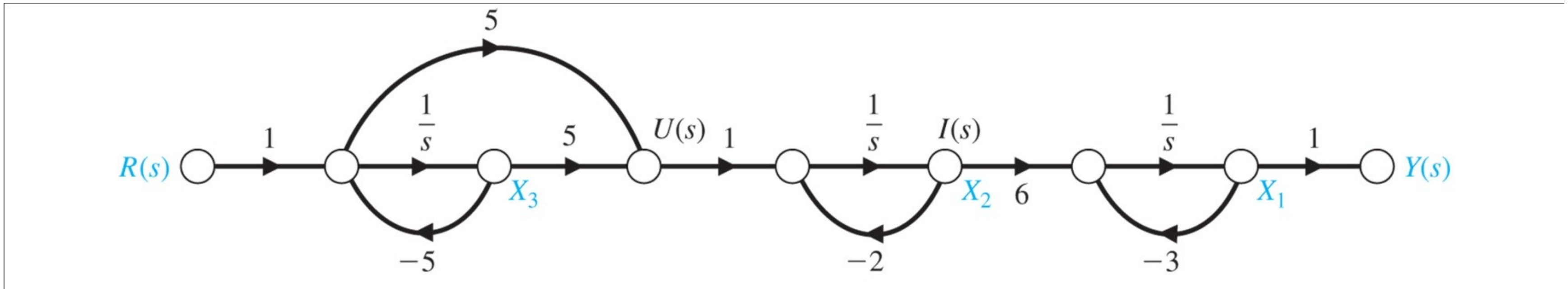


(a) The physical state variable signal-flow graph for the block diagram of Figure 3.17. (b) Physical state block diagram.

Example 24



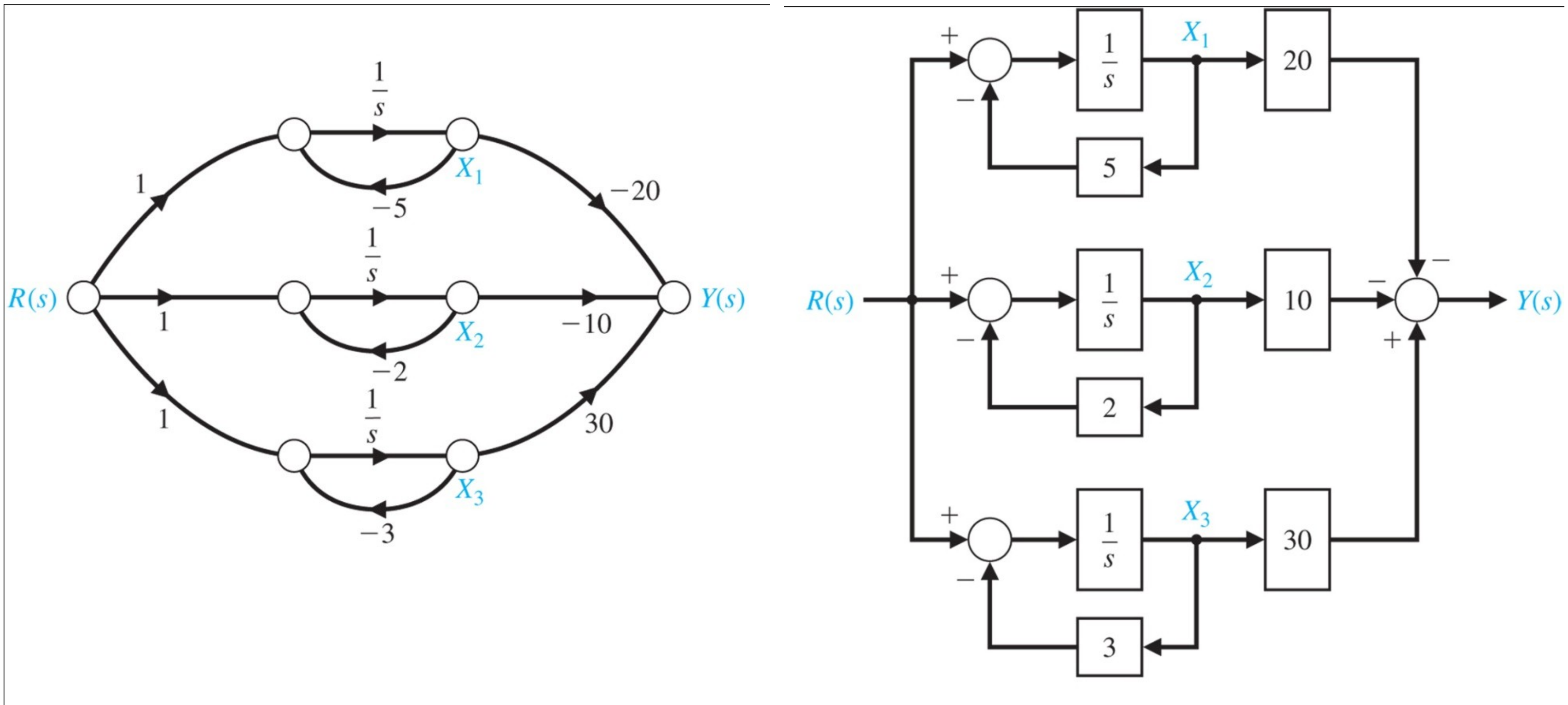
(b)



(a)

Example 25

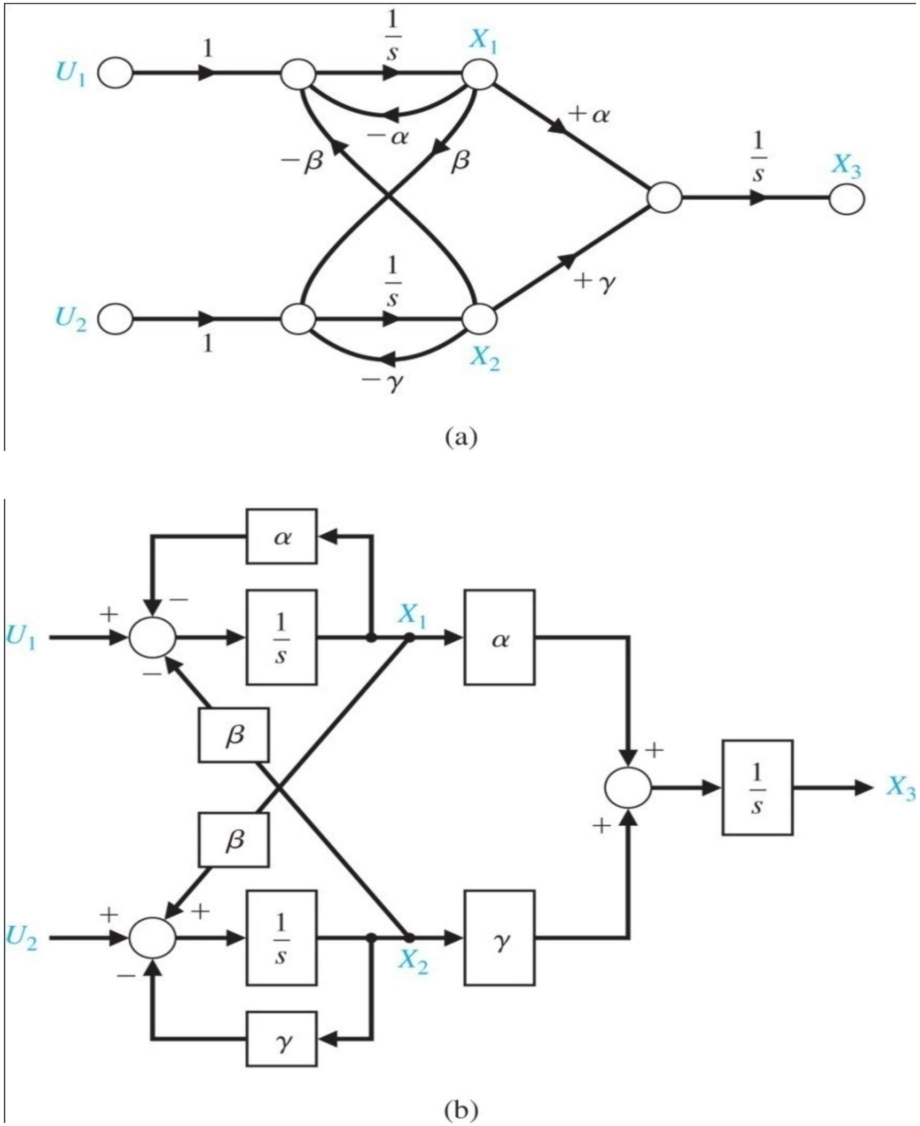
(a) The decoupled state variable flow graph model for the system shown in block diagram form in Figure 3.17. (b) The decoupled state variable block diagram model.



(a) (b)

Model for the spread of an epidemic disease. (a) Signal-flow graph. (b) Block diagram model.

Example 26



Mason's Gain Formula for SFG

Given an SFG with N forward paths and K loops, the gain between the input node y_{in} and output node y_{out} is [3]

$$M = \frac{y_{\text{out}}}{y_{\text{in}}} = \sum_{k=1}^{N} \frac{M_k \Delta_k}{\Delta} \tag{3-54}$$

where

y_{in} = input-node variable

y_{out} = output-node variable

M = gain between y_{in} and y_{out}

N = total number of forward paths between y_{in} and y_{out}

M_k = gain of the kth forward paths between y_{in} and y_{out}

$$\Delta = 1 - \sum_i L_{i1} + \sum_j L_{j2} - \sum_k L_{k3} + \dots \tag{3-55}$$

L_{mr} = gain product of the mth $(m = i, j, k, \dots)$ possible combination of r nontouching loops $(1 \le r \le K)$.

or

Δ = 1 − (sum of the gains of **all individual** loops) + (sum of products of gains of all possible combinations of **two** nontouching loops) − (sum of products of gains of all possible combinations of **three** nontouching loops) $+ \cdots$

Δ_k = the Δ for that part of the SFG that is nontouching with the kth forward path.

Mason's Gain Formula for SFG

The gain formula in Eq. (3-54) may seem formidable to use at first glance. However, Δ and Δ_k are the only terms in the formula that could be complicated if the SFG has a large number of loops and nontouching loops.

Care must be taken when applying the gain formula to ensure that it is applied between an **input node** and an **output node**.

Mason's Gain Formula for SFG

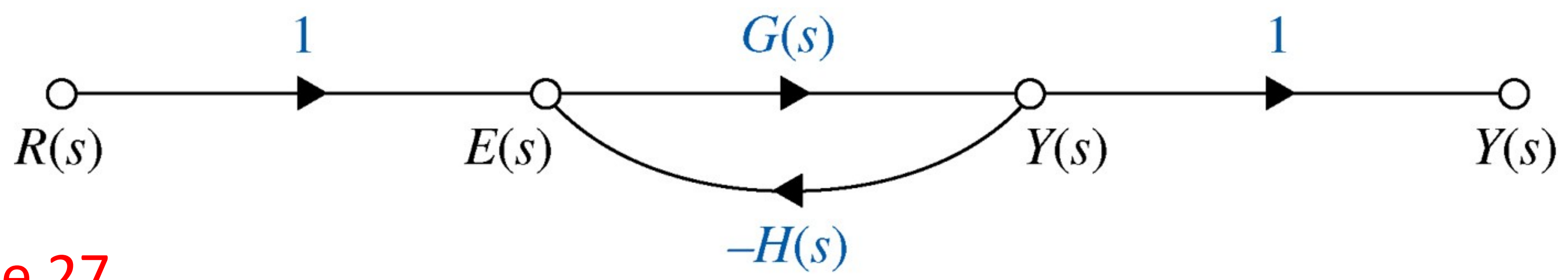


fig_03_32

Example 27

Consider that the closed-loop transfer function $Y(s)/R(s)$ of the SFG in Fig. 3-32 is to be determined by use of the gain formula, Eq. (3-54). The following results are obtained by inspection of the SFG:

1. There is only one forward path between $R(s)$ and $Y(s)$, and the forward-path gain is

$$M_1 = G(s) \tag{3-56}$$

2. There is only one loop; the loop gain is

$$L_{11} = -G(s)H(s) \tag{3-57}$$

3. There are no nontouching loops since there is only one loop. Furthermore, the forward path is in touch with the only loop. Thus, $\Delta_1 = 1$, and

$$\Delta = 1 - L_{11} = 1 + G(s)H(s) \tag{3-58}$$

Using Eq. (3-54), the closed-loop transfer function is written

$$\frac{Y(s)}{R(s)} = \frac{M_1\Delta_1}{\Delta} = \frac{G(s)}{1 + G(s)H(s)} \tag{3-59}$$

which agrees with Eq. (3-12).

Mason's Gain Formula for SFG

Consider the SFG shown in Fig. 3-25(d). Let us first determine the gain between y_1 and y_5 using the gain formula.

The three forward paths between y_1 and y_5 and the forward-path gains are

$$\begin{aligned} M_1 &= a_{12}a_{23}a_{34}a_{45} \quad \text{Forward path: } y_1 - y_2 - y_3 - y_4 - y_5 \\ M_2 &= a_{12}a_{25} \quad \text{Forward path: } y_1 - y_2 - y_5 \\ M_3 &= a_{12}a_{24}a_{45} \quad \text{Forward path: } y_1 - y_2 - y_4 - y_5 \end{aligned}$$

The four loops of the SFG are shown in Fig. 3-28. The loop gains are

$$L_{11} = a_{23}a_{32} \quad L_{21} = a_{34}a_{43} \quad L_{31} = a_{24}a_{43}a_{32} \quad L_{41} = a_{44}$$

There is only one pair of nontouching loops; that is, the two loops are

$$y_2 - y_3 - y_2 \quad \text{and} \quad y_4 - y_4$$

Thus, the product of the gains of the two nontouching loops is

$$L_{12} = a_{23}a_{32}a_{44} \tag{3-60}$$

All the loops are in touch with forward paths M_1 and M_3. Thus, $\Delta_1 = \Delta_3 = 1$. Two of the loops are not in touch with forward path M_2. These loops are $y_3 - y_4 - y_3$ and $y_4 - y_4$. Thus,

$$\Delta_2 = 1 - a_{34}a_{43} - a_{44} \tag{3-61}$$

Substituting these quantities into Eq. (3-54), we have

$$\begin{aligned} \frac{y_5}{y_1} &= \frac{M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3}{\Delta} \\ &= \frac{(a_{12}a_{23}a_{34}a_{45}) + (a_{12}a_{25})(1 - a_{34}a_{43} - a_{44}) + a_{12}a_{24}a_{45}}{1 - (a_{23}a_{32} + a_{34}a_{43} + a_{24}a_{32}a_{43} + a_{44}) + a_{23}a_{32}a_{44}} \end{aligned} \tag{3-62}$$

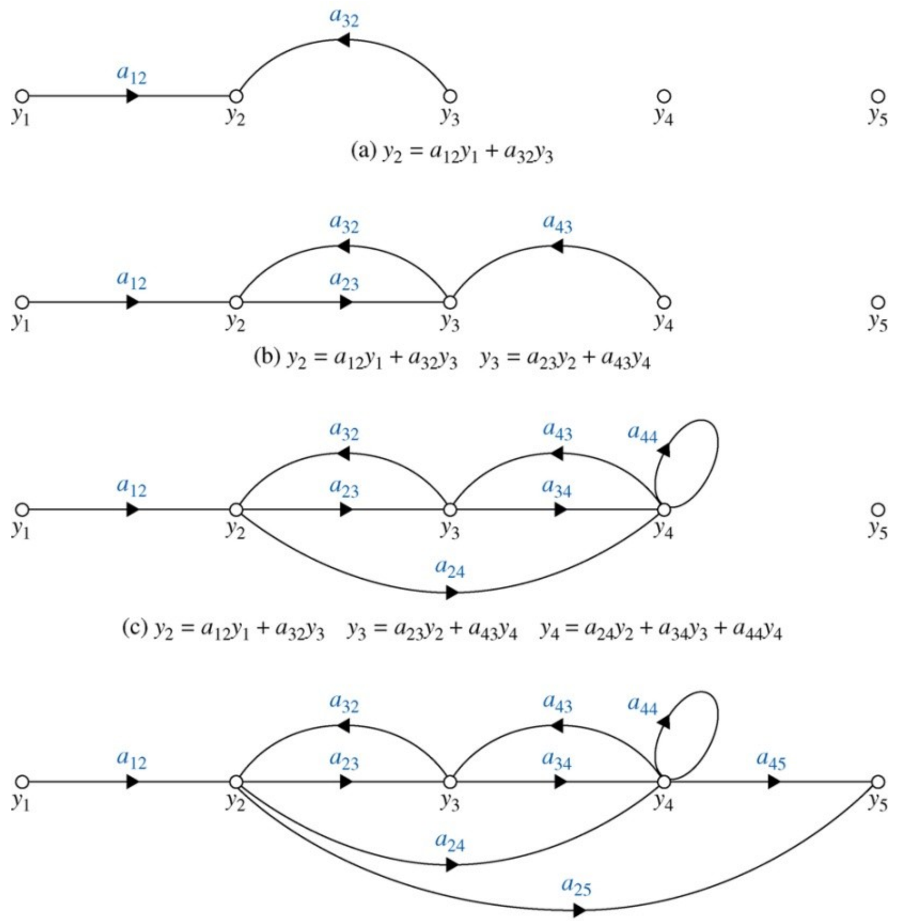
where

$$\begin{aligned} \Delta &= 1 - (L_{11} + L_{21} + L_{31} + L_{41}) + L_{12} \\ &= 1 - (a_{23}a_{32} + a_{34}a_{43} + a_{24}a_{32}a_{43} + a_{44}) + a_{23}a_{32}a_{44} \end{aligned} \tag{3-63}$$

The reader should verify that choosing y_2 as the output,

$$\frac{y_2}{y_1} = \frac{a_{12}(1 - a_{34}a_{43} - a_{44})}{\Delta} \tag{3-64}$$

where Δ is given in Eq. (3-63).



(a) $y_2 = a_{12}y_1 + a_{32}y_3$

(b) $y_2 = a_{12}y_1 + a_{32}y_3 \quad y_3 = a_{23}y_2 + a_{43}y_4$

(c) $y_2 = a_{12}y_1 + a_{32}y_3 \quad y_3 = a_{23}y_2 + a_{43}y_4 \quad y_4 = a_{24}y_2 + a_{34}y_3 + a_{44}y_4$

(d) Complete signal-flow graph

fig_03_25

Mason's Gain Formula for SFG

Consider the SFG in Fig. 3-33. The following input–output relations are obtained by use of the gain formula:

Example 28

$$\frac{y_2}{y_1} = \frac{1 + G_3H_2 + H_4 + G_3H_2H_4}{\Delta} \tag{3-65}$$

$$\frac{y_4}{y_1} = \frac{G_1G_2(1 + H_4)}{\Delta} \tag{3-66}$$

$$\frac{y_6}{y_1} = \frac{y_7}{y_1} = \frac{G_1G_2G_3G_4 + G_1G_5(1 + G_3H_2)}{\Delta} \tag{3-67}$$

where

$$\begin{aligned}\Delta = 1 &+ G_1H_1 + G_3H_2 + G_1G_2G_3H_3 + H_4 + G_1G_3H_1H_2 \\ &+ G_1H_1H_4 + G_3H_2H_4 + G_1G_2G_3H_3H_4 + G_1G_3H_1H_2H_4\end{aligned} \tag{3-68}$$

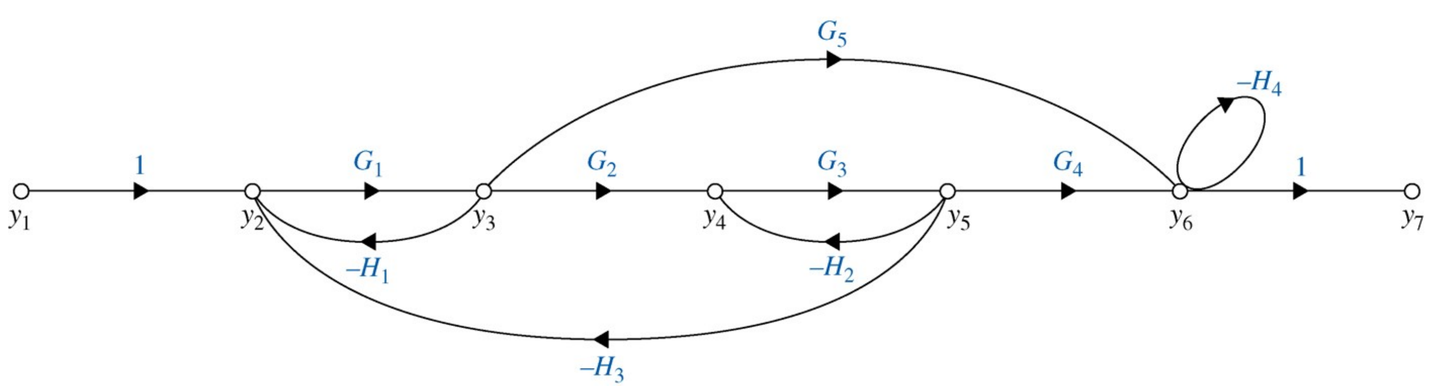


Figure 3-33 Signal-flow graph for Example 3-2-4.

Mason’s Gain Formula for SFG

Example 29

Application of the Gain Formula to Block Diagrams

$$\frac{Y(s)}{R(s)} = \frac{G_1G_2G_3 + G_1G_4}{\Delta}$$

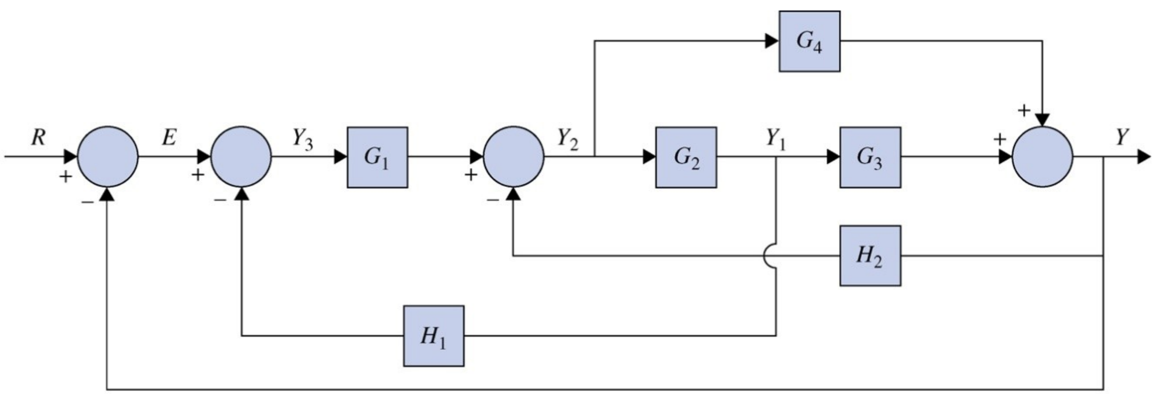
$$\Delta = 1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3 + G_4H_2 + G_1G_4$$

Forward Path Gains: 1. $G_1G_2G_3$; 2. G_1G_4

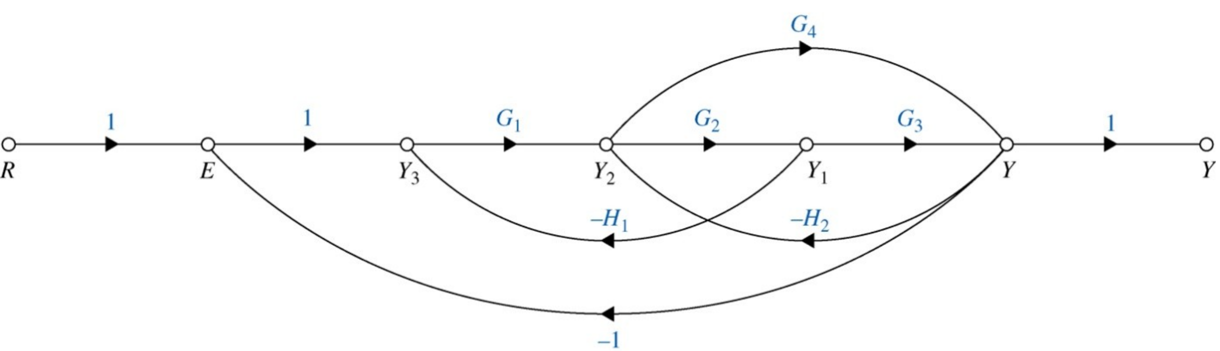
Loop Gains: 1. $-G_1G_2H_1$; 2. $-G_2G_3H_2$; 3. $-G_1G_2G_3$; 4. $-G_4H_2$; 5. $-G_1G_4$

$$\frac{E(s)}{R(s)} = \frac{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2}{\Delta}$$

$$\frac{Y(s)}{E(s)} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2}$$



(a)



(b)

Figure 3-34 (a) Block diagram of a control system. (b) Equivalent signal-flow graph.

Mason's Gain Formula for SFG

Simplified Gain Formula

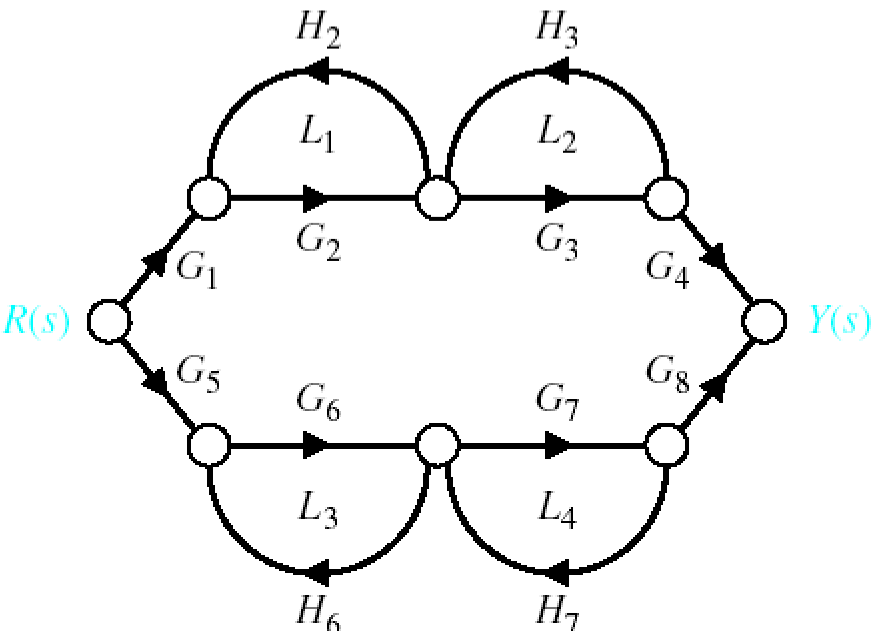
From Example 3-2-6, we can see that *all loops and forward paths are touching* in this case. As a general rule, if there are no nontouching loops and forward paths (e.g., $y_2 - y_3 - y_2$ and $y_4 - y_4$ in Example 3-2-3) in the block diagram or SFG of the system, then Eq. (3-54) takes a far simpler look, as shown next.

$$M = \frac{y_{out}}{y_{in}} = \sum \frac{\mathit{Forward\ Path\ Gains}}{1 - \mathit{Loop\ Gains}} \qquad (3\text{-}76)$$

Redo Examples 3-2-2 through 3-2-6 to confirm the validity of Eq. (3-76).

Mason's Gain Formula for SFG

Example 30



Two-path interacting system.

$$\frac{Y(s)}{R(s)} = \frac{\left[G_1 \cdot G_2 \cdot G_3 \cdot G_4 \cdot \left(1 - L_3 - L_4\right)\right] + \left[G_5 \cdot G_6 \cdot G_7 \cdot G_8 \cdot \left(1 - L_1 - L_2\right)\right]}{1 - L_1 - L_2 - L_3 - L_4 + L_1 \cdot L_3 + L_1 \cdot L_4 + L_2 \cdot L_3 + L_2 \cdot L_4}$$