Integration of Taguchi's Loss Function in the Economic Design of $\bar{x}$ - Control Charts with Increasing Failure Rate and Early Replacement

A. Al-Ghazi, K. Al-Shareef, and S. O. Duffuaa
alghazi@kfupm.edu.sa, khaled@kfupm.edu.sa, duffuaa@kfupm.edu.sa

Abstract **-In this paper, Taguchi's quadratic loss function is incorporated in the economic design of the control chart. This is done by redefining the in-control and out-of-control costs using Taguchi's loss function in the general model for the economic design of $\bar{x}$ -control charts developed by Banerjee and Rahim [1]. Both cases of increasing hazard rate (Weibull failure rate) and constant hazard rate (exponential distribution) will be presented and followed by numerical examples. Finally, sensitivity analysis is conducted to study the effect of important parameters on the cost. Suggestions for further research are also included.**

Keywords – **Taguchi methods, modeling, control charts, costs**

I. INTRODUCTION

The need for perfection and elimination of products that do not conform to specification was the main reason and motivation behind coming up with control charts. Shewart invented control charts in the last century using statistical techniques to alert production operators of a shift in production process [2]. Although control charts have been widely used since, the parameters of control charts such as sampling interval *h*, sampling size *n* and control limits *k* are still mostly determined from experience. The first economic model for $\bar{x}$ -control charts was proposed by Duncan [3]. In his model, Duncan [3] assumed that the arrivals of the assignable cause follow a Poisson process. Several extensions were made to relax the assumption to a non-Markovian process by Baker [4], Montgomery and Heikes [5] in which the design parameters (*n, k, h*) are constant over time. Lorenzen and Vance [6] extended Duncan's model for both situations when the process continues and when it is stopped during the search of the assignable causes and the repair of the out-of-control process. Duncan model motivated most of the work in the area of economic design of control chart and several survey papers appeared in the literature on this subject such as Gibra [7], Montgomery [8], and Vance [9]. Banerjee and Rahim [1] proposed a model for the design of control charts assuming increasing failure rate and variable sampling intervals. The objective in all papers related to the economic design of $\bar{x}$ -control chart is to find the optimal values for the parameters *n*, *k* and *h* that minimizes the total expected cost per unit time.

The use of control charts implies that quality loss is considered as the cost when the quality characteristics are outside the specification limits. All products falling within the control limits are considered as having the same quality regardless of the deviation of their quality characteristic from its target value. However, this is not the case when it comes to real life examples in which any deviation from the target value will incur cost to the customers. Taguchi [10,11] defined quality loss as "the loss to society caused by the product after it is shipped out". His quadratic loss function is well known and has been widely used in all fields. It is used to estimate the quality loss of a product when its quality characteristic deviates from its target value.

In recent research work an effort has been directed to incorporate the quadratic loss function in the economic design of control charts. Elsayed and Chen [12] and Alexander *et al.* [13] proposed a model for the economic design of $\bar{x}$ -control chart using quadratic loss function. These two papers didn't adequately incorporate the Taguchi philosophy in the design of control charts. Ben-Daya and Duffuaa [14] on the other hand proposed a better way to better embed the quadratic loss function in the economic design of $\bar{x}$ -control charts by having the in-control and out-of-control costs as a function of the control chart design parameters.

The purpose of this paper is to extend Taguchi's philosophy to Rahim's [15] generalized model for the economic design of $\bar{x}$ -control charts by incorporating Taguchi's quadratic loss function. The cost function of Rahim's model is modified to incorporate the

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Taguchi's loss function using the in-control and the out-of-control costs, the paper is organized as follows: the next section presents briefly Rahim's generalized model, followed by the derivation of the in-control and out-of-control costs. The following section provides the model developed in this paper, followed by illustrative examples for both varying and constant hazard function cases. The effects of each key parameter on the design parameters of the control chart are studied using sensitivity analysis. Finally, the paper is concluded in section 6.

II. RAHIM'S MODEL

In this section a brief outline of Rahim's model is provided. The model and notation, appeared in Banerjee and Rahim [1], are adopted. A brief outline of the model is given in order to make the paper self-contained.

A. Notations

n	sample size
h	length of the sampling interval
k	control limit coefficient
μ	Process mean
σ	Process standard deviation
δ	magnitude of the shift in the mean
α	Pr (exceeding control limits – process in control)
β	Pr (not exceeding control limits – process out of control)
$\phi()$	standard normal probability density function
$\Phi()$	standard normal cumulative function
Δ	acceptable deviation of the quality characteristic from its target value
A	cost of rework or scrap of a unit with quality characteristic deviating by Δ from the target
w_m	length of a production cycle up to the m^{th} sampling
$f(t)$	the probability density function of a failure distribution
$r(t)$	the hazard function
$F(t)$	the cumulative distribution function
Z_0	the expected search time associated with a false alarm
Z_1	the expected search time to discover the assignable cause and repair the process
a	fixed sampling cost
b	variable sampling cost
Y	the cost per false alarm
W	the cost to locate and repair the assignable cause
D_0	the quality cost per hour while producing in-control
D_1	the quality cost per hour while producing out-of-control
$E(T)$	expected length of a production cycle
$E(C)$	the expected cost per cycle
ECT	the expected cost per unit time
$S(x)$	the salvage value for a working equipment of age x
$\overline{F}(w_j)$	$=1-F(w_j)$
$\nabla F(w_j)$	$=F(w_j)-F(w_{j-1})$
$L_{in}(n,k)$	in control cost
$L_{out}(n,k)$	out of control cost

B. Assumptions

1- The duration of in-control period is assumed to follow an arbitrary probability density $f(t)$ having an increasing hazard rate $r(t)$.

2- The process is monitored by drawing random samples of size n at times h_1, h_1+h_2, $h_1+h_2+h_3$ … and so on. In addition, h_j satisfies:

(i) $h_1 \geq h_2 \geq h_3 \ldots$, and

(ii) $\lim_{n\to x} F(w_m)=1$. Where

$$w_m = \sum_{j=1}^{m} h_j$$

3- Production cycle ends either with a true alarm or at time w_m whichever occurs first. There is no sampling and charting during the m^{th} sampling interval.

4- Production ceases during search and repair.

Renewal of the process occurs, since the process is brought back to in control state by repair. This type of renewal process has a property that the expected cost per unit time can be expressed as the ratio of the expected cost per cycle $E(C)$ to the expected length of the cycle $E(T)$. The process will be replaced earlier at w_n if it did shift to out-of-control.

The objective of this model is to find the optimal values of *n, k,* h_1 that minimizes the total expected cost per unit time.

The expected cycle time includes:

a) In-control time during production (including work stoppage for false alarms).
b) The time between the shift to out-of-control and when the first sample point falls outside the control limits.
c) The time to search for an assignable cause and repair the machine.

$$E(T) = \sum_{j=1}^{\infty} h_j \overline{F}(w_{j-1}) + \alpha Z_0 \sum_{j=1}^{\infty} \overline{F}(w_j)$$
$$+ \beta \sum \nabla F(w_j) \sum_{i=j+1}^{\infty} h_i \beta^{i-j-1} + Z_1$$

Similarly, the expected cost per cycle consists of:

a) The cost of producing non-conforming items during in-control and out-of-control periods.
b) The cost of false alarms (including the searching and downtime).
c) The cost of locating an assignable cause and repairing the process.
d) The cost of sampling and testing.
e) The salvage value (in negative).

$$E(C) = D_0 \sum_{j=1}^{m} h_j \overline{F}(w_{j-1}) + \alpha Y \sum_{j=1}^{m-1} \overline{F}(w_j)$$
$$+ (D_o - D_1) \int_o^{w_m} x f(x) dx + (D_1 - D_0) \sum_{j=1}^{m} w_j \nabla F(w_j)$$
$$+ D_1 \beta \left[\sum_{j=1}^{m-1} \nabla F(w_j) \sum_{i=j+1}^{m} h_i \beta^{i-j-1} \right]$$
$$+ (a + bn) \left[\begin{array}{l} 1 + \sum_{j=1}^{m-2} \overline{F}(w_j) \\ + \beta \sum_{j=1}^{m-2} \nabla F(w_j) \left\{ \begin{array}{l} (1-\beta) \sum_{i=1}^{m-1-j} i \beta^{i-1} \\ + (m-1-j) \beta^{m-1-j} \end{array} \right\} \end{array} \right]$$
$$+ W - \overline{F}(w_m) S(w_m)$$

In Rahim's model [15], the authors argued that for a process with increasing hazard rate (Weibull failure rate) the length of sampling intervals shouldn't be constant as in the constant hazard rate case (exponential failure rate). So they proposed that the intervals vary with time such that the integrated hazard rate in each interval is equal. This can be shown in the following relation:

$$\int_{w_j}^{w_{j+1}} r(t) dt = \int_0^{w_1} r(t) dt \quad \text{for } j = 1,2,.....,m-1$$

which is equivalent to

$$\overline{F}(w_j) = \left[\overline{F}(w_1)\right]^j \quad \text{for } j = 1,2,.....,m$$

The economic benefits of a non-uniform sampling scheme over the uniform sampling will be shown in the numerical example.

III. TAGUCHI'S PHILOSOPHY AND THE $\overline{x}$ - CONTROL CHARTS

The main motivation behind integrating Taguchi's approach to statistical process control is to distinguish between products that fall within the specification limits. That is, a distinction should be made between a product whose quality characteristic is near the target and a product which is near either that upper or lower specification limits. Quadratic loss function was proposed by Taguchi to estimate the loss occurring as a result from deviating from the target value. So products which used to be satisfactory in the old statistical SPC may now incur different quality loss under Taguchi's approach. The in-control and out-of-control costs will be modified in the original model by Rahim so that Taguchi's offline concepts are brought to the new model.

Ben-Daya and Duffuaa [14] came up with a nice cost function for the in-control and the out-of-control production assuming the following:

- The process is monitored using $\overline{x}$ - control charts, and it's producing products with symmetric nominal type and bilateral tolerance equal to Δ.

- During in-control the process is centered at $\mu = \mu_0$ which is the target value, however during out-of-control the process mean shifts from μ to $\mu \pm \delta\sigma$.
- The process is capable; thus the tail of the normal distribution of sample means outside the specification limits can be neglected.

The in-control and out-of-control costs can be easily found after examining the figure 1.

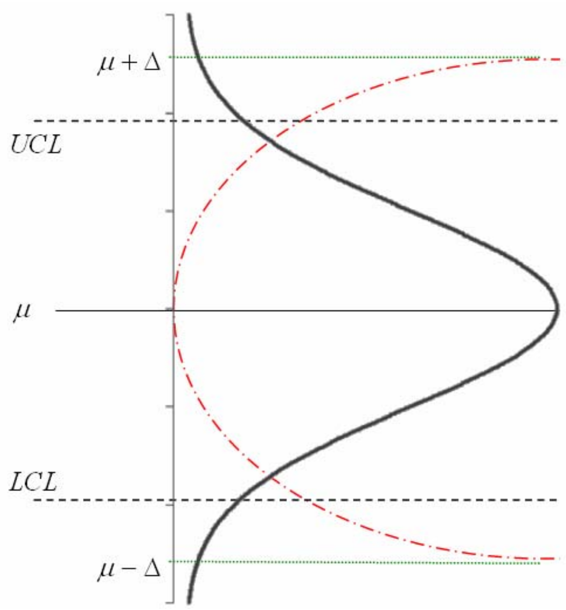

Fig.1. A diagram for the new model.

From Figure 1, the in-control cost is given by:

$$LCL = \mu - k\sigma / \sqrt{n}$$
$$UCL = \mu + k\sigma / \sqrt{n}$$
$$L_{in}(n,k) = \frac{A}{\Delta^2} \int_{\mu - k\sigma/\sqrt{n}}^{\mu + k\sigma/\sqrt{n}} (y-\mu)^2 f(y) dy$$

Where Y is a random variable denoting sample means of the quality characteristic, and *f(y)* is the normal density function with mean μ and standard deviation $\sigma/\sqrt{n}$.

It can be seen that any deviation from the target value will incur a cost even if it was within the control limits.

In the same fashion, the out-of-control cost is given by:

$$L_{out}(n,k) = \frac{A}{\Delta^2} \left[\int_{\mu-\Delta}^{\mu - k\sigma/\sqrt{n}} (y'-\mu)^2 f(y') dy' + \int_{\mu + k\sigma/\sqrt{n}}^{\mu+\Delta} (y'-\mu)^2 f(y') dy' \right]$$

Where Y' is a random variable denoting sample means of the quality characteristic, and *f(y')* is the normal density function with mean $\mu \pm \delta\sigma$ and standard deviation $\sigma/\sqrt{n}$.

Using the assumption of neglecting the tails of the normal distribution outside the specification limits, the out-of-control time can be alternatively expressed as:

$$L_{out}(n,k) = \frac{A}{\Delta^2} \left[\int_{-\infty}^{\infty} (y'-\mu)^2 f(y') dy' - \int_{\mu - k\sigma/\sqrt{n}}^{\mu + k\sigma/\sqrt{n}} (y'-\mu)^2 f(y') dy' \right]$$

By doing some algebraic manipulations, both costs can be expressed as:

$$L_{in}(k) = \frac{A\sigma^2}{n\Delta^2} \left[1 - \frac{2k}{\sqrt{2\pi}} e^{-k^2/2} - \alpha \right]$$

$$\alpha = 2\Phi(-k)$$

$$L_{out}(k) = \frac{A\sigma^2}{n\Delta^2} \left[(1+\delta^2 n)(1-\beta) + \frac{k+\delta\sqrt{n}}{\sqrt{2\pi}} e^{-(k-\delta\sqrt{n})^2/2} + \frac{k-\delta\sqrt{n}}{\sqrt{2\pi}} e^{-(k+\delta\sqrt{n})^2/2} \right]$$

$$\beta = \Phi(k - \delta\sqrt{n}) + \Phi(k + \delta\sqrt{n})$$

The modified expected cost per cycle for Rahim's model with Taguchi's approach embedded will be:

$$E(C) = PL_{in}\sum_{j=1}^{m} h_j \bar{F}(w_{j-1}) + \alpha Y \sum_{j=1}^{m-1} \bar{F}(w_j)$$

$$+ P(L_{in} - L_{out}) \int_{o}^{w_m} x f(x) dx + P(L_{out} - L_{in}) \sum_{j=1}^{m} w_j \nabla F(w_j)$$

$$+ PL_{out}\beta \left[\sum_{j=1}^{m-1} \nabla F(w_j) \sum_{i=j+1}^{m} h_i \beta^{i-j-1} \right]$$

$$+ (a+bn) \left[\begin{array}{l} 1 + \sum_{j=1}^{m-2} \bar{F}(w_j) \\ + \beta \sum_{j=1}^{m-2} \nabla F(w_j) \left\{ \begin{array}{l} (1-\beta) \sum_{i=1}^{m-1-j} i \beta^{i-1} \\ + (m-1-j)\beta^{m-1-j} \end{array} \right\} \end{array} \right]$$

$$+ W - \bar{F}(w_m) S(w_m)$$

Where P is the production rate, L_{in} is the cost while producing in the in-control period and L_{out} is the cost while producing in the out-of-control period.

The expected total cost per unit time is

$$ETC(n,k,h) = \frac{E(C)}{E(T)}$$

The problem is to find the optimal parameters n, k and h that minimizes the expected total cost per unit time.

IV. SOLUTION PROCEDURE AND EXAMPLES

In this section, numerical examples will be shown for the case of increasing hazard rate (using Weibull distribution for failure rate) with both the uniform and non-uniform sampling schemes. In addition, another example will illustrate the case of constant hazard rate (using exponential distribution for failure rate) with a uniform sampling scheme. In order to calculate the optimum values for the parameters, the authors modified the computer program which was published by Rahim in the Journal of Quality Technology [15] and used for that purpose. The program uses a search algorithm based on the pattern search technique of Hook and Jeeves [16].

Case 1: Constant hazard rate

In this case the failure rate follows an exponential distribution

$$f(t) = \lambda e^{-\lambda t} \qquad t > 0 \quad \lambda > 0$$

Example 1:

Consider the following parameters

A=\$5, $\sigma/\delta = 0.1$, P=100, $\delta = 2$, $\lambda = 0.05$, Z_0=0.25, Z_1=1, a=\$1 b=\$0.1, Y=\$25, W=\$50

- The optimal design parameters for the control chart are: n=12, h=2.1, k=3.4 with a total expected cost per unit time equal to \$3.65

Case 2: Increasing hazard rate

In this case the failure rate follows a Weibull distribution

$$f(t) = \lambda v t^{(v-1)} e^{-\lambda t^{v}} \qquad t > 0 \quad v \geq 1 \quad \lambda > 0$$

For the case of Weibull, it was shown that

$$\bar{F}(w_j) = \left[\bar{F}(w_1)\right]^j \quad \text{for } j = 1,2,.....,m$$

This can be written as

$$h_j = \left[j^{1/v} - (j-1)^{1/v} \right] h_1$$

Example 2:

Consider the parameters of example 1 but with $\lambda = 0.003, v = 3$

- The optimal design parameters for the control chart under uniform sampling scheme are: n=10, h=1.23, k=3.28 with a total expected cost per unit time equal to \$6.92
- The optimal design parameters for the control chart under non-uniform sampling scheme are: n=11, h_1=4.4 7, k= 3.26 with a total expected cost per unit time equal to \$6.41

It can be seen here that using the non-uniform scheme saved about 7.5% when it is compared to the uniform scheme cost.

V. SENSITIVITY ANALYSIS

In this section the sensitivity analysis is conducted using both the Weibull and exponential failure cases to see the effect of some parameters on the economic design of the control chart's parameters.

		n	k	H	alpha	power	ETC
A	**5**	12	3.382	2.0961	0.0007	0.9998	3.65
	10	14	3.4	1.525	0.0007	1	4.93
	20	16	3.3898	1.1172	0.0007	1	6.9
σ/δ	**0.05**	10	3.2734	4.5055	0.0011	0.9989	2.21
	0.1	12	3.382	2.0961	0.0007	0.9998	3.65
	0.5	27	3.2781	0.5352	0.0010	1	18
δ	**1.5**	15	3.143	3.043	0.0017	0.9962	3.16
	2	12	3.382	2.0961	0.0007	0.9998	3.65
	2.5	10	3.5055	1.5875	0.0005	1	4.12
λ	**0.01**	17	3.3984	5.0937	0.0007	1	1.61
	0.05	12	3.382	2.0961	0.0007	0.9998	3.65
	0.5	9	3.1484	0.7672	0.0016	0.9978	12.34

Table 1: Effects of key parameters on the design for the exponential case

		n	k	h	alpha	power	ETC
A	**5**	11	3.2586	4.468	0.0011	0.9996	6.41
	10	12	3.3117	3.8875	0.0009	0.9999	8.11
	20	14	3.3398	3.4531	0.0008	1	10.68
σ/δ	**0.06**	11	3.0250	0.9672	0.0025	0.9998	4.81
	0.1	11	3.2586	4.468	0.0011	0.9996	6.41
	0.6	23	3.243	2.4578	0.0012	1	32.12
δ	**1.5**	15	2.9875	5.3266	0.0028	0.9976	5.68
	2	11	3.2586	4.468	0.0011	0.9996	6.41
	2.5	9	3.4117	3.9898	0.0006	1	7.09
$v = 2, \lambda$							
	0.0001	18	3.3867	24.5195	0.0007	1	1.64
	0.003	12	3.3492	6.6484	0.0008	0.9998	3.93
	0.05	10	3.2266	2.4438	0.0013	0.999	8.51
$v = 3, \lambda$							
	0.0001	13	3.3375	11.1992	0.0008	0.9999	3.4
	0.003	11	3.2586	4.468	0.0011	0.9996	6.41
	0.05	10	3.1156	2.1617	0.0018	0.9993	10.55

Table 2: Effects of key parameters on the design for the Weibull case with non-uniform sampling scheme

From tables 1 and 2, it can be shown clearly that increasing A will lead to an increase in n, k and ETC in both cases. However increasing A will lead to a small decrease in h, and the control limits become tighter while sampling is done more often because of the high scrap cost. In addition, increasing the ratio of σ/δ will lead to an increase in n and ETC but a decrease in h. For the δ parameter, increasing it will lead to a decrease in n and h but an increase in k and ETC so it becomes easier to detect the out-of-control state. Moreover, increasing λ for the exponential case will lead to a decrease in n and h but an increase in ETC because the frequency of shift to out-of-control increases and thus closer monitoring is required. Similarly for the Weibull, increasing λ will have the same effect and increasing v will lead to an increase in ETC and a decrease in n, h and k.

VI. CONCLUSION

In this paper, Taguchi's loss function was incorporated into Rahim's economic design of $\bar{x}$-control charts by redefining the in-control and the out-of-control costs. The resulting model combines the advantages of the economic design and Taguchi's philosophy. Examples are given to illustrate the idea and sensitivity analysis was done. The model can be extended to situations where other types of Taguchi's loss function are more appropriate.

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REFERENCES

[1] M. A. Rahim and P. K. Banerjee, "A generalized model for the economic design of $\bar{x}$-control charts for production systems with increasing failure rate and early replacement", *Naval Research Logistics*, Vol. 40, pp. 787-809, 1993.

[2] D. C. Montgomery, *Introduction to Statistical Quality Control*, John Wiley, New York, NY, 1993.

[3] A. J. Duncan, "The economic design of $\bar{x}$-charts used to maintain current control of a process", *Journal of the American Statistical Association*, Vol. 51, pp. 228-242, 1956.

[4] K. R. Baker, "Two models in the economic design of an $\bar{x}$ - chart", *AIIE Transactions*, Vol. 3, pp. 257-263, 1971.

[5] D. C. Montgomery, and R. G. Heikes, "Process Failure Mechanisms and Optimal Design of Fraction Defective Control Charts", *AIIE Transactions*, Vol. 8, pp. 467-474, 1976.

[6] T. J. Lorenzen, and L. C. Vance, "The Economic Design of Control Charts: A Unified Approach" *Technometrics*, Vol. 28, pp. 3-10, 1986.

[7] I. N. Gibra, "Recent developments in control charts techniques", *Journal of Quality Technology*, Vol. 7, pp. 183-192, 1975.

[8] D. C. Montgomery, "The economic design of control charts: a review and literature survey", *Journal of Quality Technology*, Vol. 12, pp. 75-87, 1980.

[9] L. C. Vance, "A bibliography of statistical quality control chart techniques, 1970-1980", *Journal of Quality Technology*, Vol. 15, pp. 59-62, 1983.

[10] G. Taguchi, "The role of metrological control for quality control", *Proceedings of the International Symposium on Metrology for Quality Control in Production*, pp. 1-7, 1984.

[11] G. Taguchi, E. A. Elsayed, and T. Hsiang, Quality Engineering in Production Systems, *McGraw-Hill*, New York, NY, 1989.

[12] E. A. Elsayed, and A. Chen, "An economic design of $\bar{x}$-control chart using quadratic loss function", *International Journal of Production Research*, Vol. 32, pp. 873-887, 1994.

[13] S. M. Alexander, M. A. Dillman, J. S. Usher and B. Damodaran, "Economic design of control charts using the Taguchi loss function", *Computers and Industrial Engineering*, Vol. 28, pp. 671-679, 1995.

[14] S. O. Duffuaa and M. Ben-Daya, "Integration of Taguchi's loss function approach in the economic design of $\bar{x}$-chart ", *International Journal of Quality & Reliability Management*, Vol. 20 No. 5, pp. 607-619, 2003.

[15] Rahim, "Economic design of $\bar{x}$-control charts assuming Weibull in-control times", *Journal of Quality Technology*, Vol.25, No.4, 1993.

[16] R. Hooke and T. A. Jeeves, "Direct search solution of numerical and statistical problems", *Journal of the Association for Computing Machinery*, Vol. 8, pp. 212-229, 1961.