



The distribution-free newsboy problem: Extensions to the shortage penalty case

Hesham K. Alfares^{*}, Hassan H. Elmorra

Systems Engineering Department, P.O. Box 5067, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

Abstract

In the classical newsboy problem, no cost is assumed if the ordered quantity is less than the demand. However, in reality failure to meet demand is always associated with a penalty. The aim of this work is to extend the analysis of the distribution-free newsboy problem to the case when shortage cost is taken into consideration. The analysis is based on the assumption that only the mean and variance of demand are known, but its particular probability distribution is not. A model is presented for determining both an optimal order quantity and a lower bound on the profit under the worst possible distribution of the demand. The following cases are considered: the single product case, the fixed ordering cost case, the random yield case, and the resource-constrained multi-product case.

© 2004 Elsevier B.V. All rights reserved.

Keywords: Newsboy problem; Distribution free approach; Single-period inventory systems; Lot sizing

1. Introduction

The classical newsboy problem aims to find the order quantity for a given product that maximizes the expected profit in a single period, probabilistic demand framework. Gallego and Moon (1993) define the newsboy problem as the tool to decide the stock quantity of an item when there is a single purchasing opportunity before the start of the selling period, and the demand for the item is random. The classical newsboy model assumes that if the order quantity is larger than the realized demand, a single discount is used to sell excess inventory or that excess inventory is disposed off. On the other hand, if the order quantity is less than demand, then profit is lost. The objective is to find the optimum tradeoff between the risk of overstocking (incurring disposal cost) and the risk of under stocking (losing profit).

The newsboy problem is a classical inventory problem that is very significant in terms of both theoretical and practical considerations. Items that can be classified under single-period inventory systems include

^{*}Corresponding author. Tel.: +9663-860-3726; fax: +9663-860-2965.
E-mail address: hesham@ccse.kfupm.edu.sa (H.K. Alfares).

Christmas trees, new-year greeting cards, and of course daily newspapers. The newsboy problem is often used to aid decision-making in fashion, sporting industry, and apparel industry, both at the manufacturing and retailer level (Gallego and Moon, 1993). According to Weatherford and Pfeifer (1994), newsboy models are also used in capacity management and booking decisions in service industries such as hotels and airlines.

Several models and extensions of the classical newsboy problem have been proposed and solved in the literature. Khouja (1999) classifies solution approaches into two types; the first minimizes expected cost, while the second maximizes expected profit. Khouja (1999) also classifies extensions to the basic newsboy problem into 11 categories, including: random yields where orders received may include defective items, different states of information about demand, and resource-constrained multiple products. This paper addresses these three extensions, in addition to the fixed ordering cost case. However, the main focus of this paper is on the extension based on limited information about demand, namely the distribution-free newsboy problem.

Several authors have analyzed the distribution-free newsboy problem, in which the distribution of demand is not known but only the mean μ and variance σ^2 are specified. Usually, a minimax approach is followed, which aims to minimize the maximum cost resulting from the worst possible demand distribution. Scarf (1958), who pioneered this approach, uses it to develop a closed form expression for the order quantity that maximizes expected profit. Kasugai and Kasegai (1960, 1961), respectively, apply dynamic programming and the minimax regret ordering policy to the distribution-free multi-period newsboy problem.

Gallego and Moon (1993) provide a simpler proof of optimality of Scarf's ordering rule and extend the analysis to the cases of random yields, fixed ordering cost, and constrained multiple products, in addition to the recourse case where there is a second ordering opportunity. Moon and Choi (1995) extend the model of Gallego and Moon (1993) to the case in which customers may balk if the inventory level is low. Moon and Choi (1997) subsequently use a similar approach to analyze a newsboy problem with various degrees of product processing, from raw materials to the finished product. The alternative policies in this case include make-to-order, make-in-advance, and composite policies, with and without budget limitations.

Vairaktarakis (2000) develops several minimax regret models for the distribution-free multi-item newsboy problem under a budget constraint and two types of demand uncertainty. The interval type specifies a lower bound and an upper bound on demand, while the discrete type states a set of likely demand values. Moon and Silver (2000) develop distribution-free models and heuristics for a multi-item newsboy problem with a budget constraint and fixed ordering costs. Recent and comprehensive literature reviews and suggestions for future research on the newsboy problem are compiled by Khouja (1999), Petruzzi and Dada (1999), and Silver et al. (1998).

The model of Gallego and Moon (1993), which constitutes the core background of this work, provides the optimal order quantity that maximizes the expected profit against the worst possible distribution of the demand with mean μ and variance σ^2 . It also presents a simple lower bound on the expected profit with respect to all possible distributions of demand. The purpose of this work is to extend the optimal order quantity formulas found by Gallego and Moon (1993) to the case where shortage cost (above and beyond lost profit) is considered for the following cases: the single product case, the fixed ordering cost case, the random yield case, and the multi-product case. Since the recourse case considered by Gallego and Moon (1993) rules out shortages, it is not considered in this paper. Although the shortage (lost sales) cost is generally not easy to estimate, it is nonetheless a real cost that should not be simply ignored. Including any reasonable estimate of this cost significantly improves the accuracy and profitability of the newsboy model. Numerical experiments are presented to demonstrate the profit increase obtained from the new models that incorporate shortage cost.

This paper is organized as follows. In Section 2, the optimum order quantity and the lower bound on expected profit are derived for the single product case. The fixed ordering cost case is analyzed in Section 3,

while the random yield case is considered in Section 4. In Section 5, the multi-product case with a budget constraint is investigated. Numerical experiments are presented in Section 6. Finally, conclusions and suggestions for future research are given in Section 7.

2. Single product case

The aim of this work is to extend the distribution free models found in Gallego and Moon (1993) to the case where a shortage penalty cost is considered. Therefore, we are going to adapt the notation used in Gallego and Moon (1993). Let D denote the random demand, whose unknown distribution G belongs to the class \mathcal{G} of cumulative distribution functions with mean μ and variance σ^2 . First, a basic model is developed for the single product. The notation needed for the model is given as

- $c > 0$ unit cost
- $p = (1 + m)c > c$ unit selling price
- $s = (1 - d)c < c$ unit salvage value
- μ expected demand
- σ standard deviation of demand
- Q order quantity
- m mark-up, i.e., return per dollar on units sold
- d discount, i.e., loss per dollar on unit unsold
- D random demand
- l lost sales (shortage) cost per unit, beyond lost profit $p - c$
- $k = \frac{l}{c} < c$ fractional unit shortage cost
- $x^+ = \max\{x, 0\}$ the positive part of x

2.1. Optimal order quantity

The expected profit expression is given in Gallego and Moon (1993) as

$$\pi^G(Q) = pE(\min\{Q, D\}) + sE(Q - D)^+ - cQ. \tag{1}$$

In (1), there is no cost when the ordered quantity is less than the demand. However in reality any failure to meet demand is always associated with a penalty. Thus, we define a new profit expression, which takes this penalty into consideration as follows:

$$\pi^G(Q) = pE(\min\{Q, D\}) + sE(Q - D)^+ - cQ - lE(D - Q)^+. \tag{2}$$

In this study the sale is considered to be of short duration, thus there is no opportunity to reorder, i.e., there is no recourse. Substituting the definitions of l , p , and s , and using the following relationships:

$$\begin{aligned} \min\{Q, D\} &= D - (D - Q)^+, \\ (Q - D)^+ &= (Q - D) + (D - Q)^+, \\ (D - Q)^+ &= (D - Q) + (Q - D)^+. \end{aligned}$$

Eq. (2) can be written as

$$\pi^G(Q) = c[(m + d - k)\mu - (d - k)Q - (d + m)E(D - Q)^+ - kE(Q - D)^+]. \tag{3}$$

In order to maximize (3) we need the following lemmas from Gallego and Moon (1993):

Lemma 1.

$$E(D - Q)^+ \leq \frac{[\sigma^2 + (Q - u)^2]^{1/2} - (Q - \mu)}{2}. \tag{4}$$

Lemma 2.

$$E(Q - D)^+ \leq \frac{[\sigma^2 + (\mu - Q)^2]^{1/2} - (\mu - Q)}{2}. \tag{5}$$

Lemma 3. For every Q , there exist a distribution $G^* \in \mathcal{G}$ where the bounds (4) and (5) are tight.

Proof of Lemma 1 is given in Gallego and Moon (1993) and Gallego (1992), but a similar proof can be easily developed for Lemma 2. Similarly, proof of Lemma 3, which is given in Gallego and Moon (1993) and Gallego (1992) for the bound (4), can be easily adapted for the bound (5). Using the three lemmas, the lower bound on expected profit $\pi^G(Q)$ can be written as

$$\begin{aligned} \pi^G(Q) \geq c & \left[(m + d - k)\mu - (d - k)Q - (d + m) \frac{[\sigma^2 + (Q - u)^2]^{1/2} - (Q - u)}{2} \right. \\ & \left. - k \frac{[\sigma^2 + (\mu - Q)^2]^{1/2} - (\mu - Q)}{2} \right]. \end{aligned} \tag{6}$$

Obviously maximizing the lower bound on expected profit (6) is equivalent to minimizing the following function:

$$\begin{aligned} \nabla\pi(Q) &= (d - k)Q + (d + m) \frac{[\sigma^2 + (Q - u)^2]^{1/2} - (Q - \mu)}{2} + k \frac{[\sigma^2 + (\mu - Q)^2]^{1/2} - (\mu - Q)}{2}, \\ \nabla\pi(Q) &= \frac{1}{2} \{ (d - m - k)Q + (d + m + k)[\sigma^2 + (Q - \mu)^2]^{1/2} + (d + m - k)\mu \}. \end{aligned} \tag{7}$$

Our problem now is to minimize $\nabla\pi(Q)$. As will be shown later, (7) is convex in Q . Thus in order to minimize the right-hand side of (7), we compute its first derivative with respect to Q and set the derivative equal to zero. Solving for Q , we obtain

$$Q^* = \mu + \frac{\sigma b}{[1 - b^2]^{1/2}}, \quad \text{where } b = \frac{k + m - d}{k + m + d}$$

or

$$Q^* = \mu + \frac{\sigma}{2} \left[\left(\frac{k + m}{d} \right)^{1/2} - \left(\frac{d}{k + m} \right)^{1/2} \right]. \tag{8}$$

To verify that (7) is strictly convex in Q , we only need to compute the second derivative of (7) with respect to Q :

$$\frac{(d + m + k)\sigma^2}{2[\sigma^2 + (Q - \mu)^2]^{1/2}} > 0.$$

Since the second derivative has the sign of $(d + m + k)/2$, which is strictly > 0 , (7) is strictly convex in Q . Therefore, (8) minimizes (7), and consequently maximizes (2) against the worst possible distribution of demand when a shortage penalty is applied.

It is worth observing that in this case, since $k = l/c$, the order quantity Q is dependent on the unit cost c . Also notice that:

- (8) calls for an order larger than the expected demand if and only if $(k + m)/d > 1$,
- (8) calls for an order smaller than the expected demand if and only if $(k + m)/d < 1$.

If $k = 0$, i.e., no shortage penalty is considered, then the order size is equal to Q^* computed by Gallego and Moon (1993). Moreover, in the typical formulation where the salvage value is zero ($d = 1$), the optimal order size is

- larger than the expected demand if and only if the quantity $(k + m)$ is larger than 1,
- smaller than the expected demand if and only if the quantity $(k + m)$ is smaller than 1.

2.2. The lower bound on expected profit

In order to calculate the lower bound on expected profit, we substitute (7) and (8) into (6), obtaining

$$\pi^G(Q) \geq mc\mu \left[1 - \frac{\sigma}{m\mu} (kd + md)^{1/2} \right]. \tag{9}$$

The following remarks can be made about the bound specified by (9):

- The lower bound is linearly increasing (decreasing) in $\mu(\sigma)$ and convexly increasing (decreasing) in $m(d)$.
- If there is no shortage penalty, i.e., $k = 0$, (9) reduces to the lower bound found by Gallego and Moon (1993):

$$\pi^G(Q) \geq mc\mu \left[1 - \frac{\sigma}{\mu} \left(\frac{d}{m} \right)^{1/2} \right]. \tag{10}$$

- The upper limit on $\pi^G(Q)$ corresponds to the maximum expected profit, which is obtained when demand is deterministic, that is, when $\sigma = 0$. This upper bound is given by

$$\pi^G(Q) \leq mc\mu. \tag{11}$$

- When demand is probabilistic ($\sigma > 0$), the associated profit loss can be considered as the fractional cost of randomness, which is given by

$$\frac{\sigma}{m\mu} (kd + md)^{1/2}. \tag{12}$$

2.3. The ordering rule

An order of size Q^* should be made only if the expected profit is not negative, i.e., the right-hand side of (9) is positive.

$$\left(\frac{m\mu}{\sigma} \right)^2 > (kd + md). \tag{13}$$

Thus, the optimal ordering rule is given as follows:

- order $Q = Q^*$ units if (13) holds, and
- order $Q = 0$ units otherwise.

Example 1. This example is adapted from Gallego and Moon (1993). The mean and the standard deviation of the demand for a given product are: $\mu = 900$ and $\sigma = 122$, respectively. Other relevant data are:

Unit cost: $c = \$35.10$,

Unit selling price: $p = \$50.30$,

Unit salvage value: $s = \$25.00$,

Unit shortage cost: $l = \$14.00$.

We will compute both Q^* and $\pi^G(Q)$, with and without shortage cost.

First, we compute m , d , and k :

$$m = p/c - 1 = 0.433,$$

$$d = 1 - s/c = 0.288,$$

$$k = l/c = 0.399.$$

Without shortage cost, we assume $k = l/c = 0$, and use (8), (9), and (11) to obtain:

$$Q^* = 925,$$

$$\$12,168 \leq \pi^G(Q) \leq \$13,680.$$

With shortage cost, (8), (9), and (11) yield

$$Q_s^* = 968,$$

$$\$11,585 \leq \pi^G(Q) \leq \$13,680.$$

We observe that incorporating shortage penalty cost decreases the lower bound on expected profit, i.e., we expect to lose profit. On the other hand, the order quantity increases in order provide higher protection against shortages.

3. The fixed ordering cost case

Assume now that fixed ordering cost A is charged for each order of any size $Q > 0$. This is typically referred to as the fixed charge problem. It is assumed that an (s, S) inventory replenishment policy will be followed. Let us first define the following new notation needed for this model.

A constant cost for placing an order, regardless of the order size,

I initial inventory (prior to placing the order),

$S = I + Q \geq I$ end inventory (after receiving the order)

s reorder level, i.e., minimum inventory level, below which an order must be made

Now, the minimum of S and D units are sold, $S - D$ units are salvaged, $D - S$ units are short, but only Q units are purchased. Thus, the expected profit expression (2) is now written as

$$\pi^G(S) = pE(\min\{S, D\}) + sE(S - D)^+ - c(S - I) - lE(D - S)^+ - A\mathbf{1}_{[S > I]}. \quad (14)$$

Thus, similar to expressing (7) in terms of S , maximizing (14) is equivalent to minimizing

$$\nabla\pi(S) = A\mathbf{1}_{[S > I]} + c\{-I + (d - k)S + (d + m)E(D - S)^+ + kE(S - D)^+\}, \quad (15)$$

where

$$\mathbf{1} = \begin{cases} 1 & \text{if } S > I, \\ 0 & \text{otherwise.} \end{cases}$$

Applying Lemmas 1–3, our problem now is to minimize the upper bound on $\nabla\pi(S)$:

$$\nabla\pi(S) \leq A\mathbf{1}_{[S > I]} + J(S), \tag{16}$$

where

$$J(S) = c \left\{ -I + \left(\frac{d - m - k}{2} \right) S + \frac{(d + m + k)}{2} [\sigma^2 + (S - \mu)^2]^{1/2} + \left(\frac{d + m - k}{2} \right) \mu \right\}. \tag{17}$$

To minimize $J(S)$, we set the derivative equal to zero and solve for S , obtaining

$$S^* = \mu + \frac{\sigma b}{[1 - b^2]^{1/2}}, \quad b = \frac{k + m - d}{k + m + d}$$

or

$$S^* = \mu + \frac{\sigma}{2} \left[\left(\frac{k + m}{d} \right)^{1/2} - \left(\frac{d}{k + m} \right)^{1/2} \right]. \tag{18}$$

Notice that the end inventory level S^* for the fixed ordering cost model is equal to the optimum order quantity Q^* for the single-item model. Now, substituting (18) into (17) gives

$$J(S^*) = c \{ -I - m\mu + \sigma(md + kd)^{1/2} \}. \tag{19}$$

Considering (15), the worst case expected cost is equal to

$$\begin{aligned} \nabla\pi(S) &= J(S^*) + A && \text{if an order is made,} \\ &= J(I) && \text{if an order is not made.} \end{aligned}$$

In order to determine the reorder level s , set $J(s) = A + J(S^*)$ and solve for s . Solving the equation, we obtain

$$s = \mu + \frac{(m + k - d)Y - (m + k + d)[Y^2 - (md + kd)\sigma^2]^{1/2}}{2(md + kd)}, \tag{20}$$

where

$$Y = \sigma(md + kd)^{1/2} + \frac{A}{C}. \tag{21}$$

Example 2. Using the data of Example 1, we assume that the ordering cost is given by $A = \$500$. Using (20) and (21), we obtain the reorder level

$$s^* = 882.$$

The order-up-to level S^* corresponds to Q^* , which was determined in Example 1 using (8) as

$$S^* = 968.$$

4. The random yield case

Let us now assume that the quantity ordered or produced Q is not perfect. Thus out of Q units, only $G(Q)$ units are good, where $G(Q)$ is a random variable. Let us also assume that each unit ordered or produced has the same probability ρ , of being good. Thus, $G(Q)$ is a binomial random variable with mean $Q\rho$. We now need to define

- $G(Q)$ number of good units out of Q , binomial random variable
- ρ probability for each unit of Q of being good
- $Q\rho$ mean of $G(Q)$
- $Q\rho q$ variance of $G(Q)$, $q = 1 - \rho$
- $\hat{c} = c/\rho$ expected unit cost
- $p = (1 + m)\hat{c}$ unit selling price $= (1 + m)c/\rho$
- $s = (1 - d)\hat{c}$ unit salvage value $= (1 - d)c/\rho$
- $k = l/\hat{c}$ fractional unit shortage cost $= l\rho/c$

The expected profit expression (2) becomes:

$$\pi^G(Q) = pE[\min\{G(Q), D\}] + sE[G(Q) - D]^+ - cQ - lE[D - G(Q)]^+. \tag{22}$$

Eq. (22) is simplified using the definitions of m and d . Next, instead of maximizing expected profit, we find it easier to minimize the expected cost

$$\nabla\pi(Q) = (d - k)Q + ((d + m)/\rho)E[D - G(Q)]^+ + (k/\rho)E[G(Q) - D]^+. \tag{23}$$

Applying lemmas from Gallego and Moon (1993):

$$E[D - G(Q)]^+ \leq \frac{[\sigma^2 + Q\rho q + (\rho Q - \mu)^2]^{1/2} - (\rho Q - \mu)}{2},$$

$$E[G(Q) - D]^+ \leq \frac{[\sigma^2 + Q\rho q + (\mu - \rho Q)^2]^{1/2} - (\mu - \rho Q)}{2}.$$

We substitute into (23) to obtain the upper bound on expected cost

$$\begin{aligned} \nabla\pi(Q) \leq & (d - k)Q + [(d + m)/2\rho]\{[\sigma^2 + Q\rho q + (\rho Q - \mu)^2]^{1/2} - (\rho Q - \mu)\} \\ & + (k/2\rho)\{[\sigma^2 + Q\rho q + (\mu - \rho Q)^2]^{1/2} - (\mu - \rho Q)\}. \end{aligned} \tag{24}$$

In order to minimize the upper bound, we differentiate (24) and set the derivative equal to zero. Solving for Q , we obtain

$$Q = \frac{(2\mu - q)\rho + [(2\mu - q)^2\rho^2 - 4\rho^2 X]^{1/2}}{2\rho^2}, \tag{25}$$

where

$$X = \mu^2 - \frac{4\sigma^2(k + m - d)^2 + (4q\mu - q^2)(k + m + d)^2}{16(kd + md)}. \tag{26}$$

Example 3. Using the data of Example 1, we assume that for each unit of Q , the probability of being good is $\rho = 0.9$.

Using (25) and (26), we obtain the order quantity

$$Q^* = 1076.$$

The order quantity based on perfect quality, calculated in Example 1, is 968. Naturally, the order quantity increases in order to provide protection against defective items.

5. The resource-constrained multi-product case

We now consider a multi-item newsboy problem with either a budget or a capacity constraint. For each item, the order quantity Q must be either purchased or manufactured. In the case of purchasing, a limited budget must be allocated among competing items. In the case of manufacturing, the constraint is imposed due to limited production capacity. According to Johnson and Montgomery (1974), this problem is sometimes called the stochastic product mix problem. The notation needed for this case is similar to the notation used for the single item case, but a subscript i is used to indicate item i , $i = 1, \dots, N$. For example:

- N number of items
- i item i , $i = 1, \dots, N$
- $c_i > 0$ unit cost of item i

Suppose that the cost of purchasing all the items cannot exceed a predetermined budget B . We want to find the order quantities (Q_1, \dots, Q_N) that:

- maximize the expected profit against the worst possible distribution of the demand,
- do not exceed the budget constraint, and
- take in consideration the shortage cost.

The objective now becomes the minimization of the upper bound on the total cost of all items, which is obtained by multiplying (7) by individual item cost c_i , and then summing the costs of all items.

$$\min_{Q_1, \dots, Q_N} \sum_{i=1}^N c_i \left\{ (d_i - k_i)Q_i + \frac{(d_i + m_i + k_i)}{2} [\sigma_i^2 + (Q_i - \mu_i)^2]^{1/2} + \frac{k_i - d_i - m_i}{2} (Q_i - \mu_i) \right\}. \tag{27}$$

This objective is optimized under the following budget constraint:

$$\sum_{i=1}^N c_i Q_i \leq B. \tag{28}$$

A convenient way to solve this constrained optimization problem is to use the Lagrange multiplier approach. In order to convert the problem into unconstrained optimization form, we construct the Lagrangian function

$$L(Q_1, \dots, Q_N, \lambda) = \sum_{i=1}^N c_i \left\{ (d_i - k_i)Q_i + \frac{(d_i + m_i + k_i)}{2} [\sigma_i^2 + (Q_i - \mu_i)^2]^{1/2} + \frac{k_i - d_i - m_i}{2} (Q_i - \mu_i) \right\} + \lambda \left(\sum_{i=1}^N c_i Q_i - B \right), \tag{29}$$

where λ is the Lagrange multiplier associated with the budget constraint.

In order to minimize L , we set the partial derivative of L with respect to each Q_i equal to zero. Solving for Q_1, \dots, Q_N , we obtain

$$Q_i = \mu_i + \frac{\sigma_i b_i}{(1 - b_i^2)^{1/2}}, \quad \text{where} \quad b_i = \frac{m_i + k_i - d_i - 2\lambda}{d_i + m_i + k_i}, \quad i = 1, \dots, N,$$

or

$$Q_i = \mu_i + \frac{\sigma_i}{2} \left[\left(\frac{m_i + k_i - \lambda}{d_i + \lambda} \right)^{1/2} - \left(\frac{d_i + \lambda}{m_i + k_i - \lambda} \right)^{1/2} \right], \quad i = 1, \dots, N. \tag{30}$$

5.1. Lower bound on expected profit

In order to calculate the lower bound on expected profit, we substitute (30) into (27) to obtain the lower bound for individual item i , and then sum the bounds for all items:

$$\pi^G(Q) \geq \sum_{i=1}^N m_i c_i \mu_i \left[1 - \frac{\sigma_i}{2m_i \mu_i} \left(\frac{(2d_i + \lambda)(k_i + m_i) - d_i \lambda}{[(k_i + m_i - \lambda)(d_i + \lambda)]^{1/2}} \right) \right]. \tag{31}$$

Obviously, the upper bound on expected profit is obtained if the demand is deterministic, i.e., $\sigma_i = 0$ for $i = 1, \dots, N$. This upper bound is given by

$$\pi^G(Q) \leq \sum_{i=1}^N m_i c_i \mu_i. \tag{32}$$

5.2. Solution procedure

To make a positive order of item i , expected profit must be positive, i.e., $\pi^G(Q_i) \geq 0$, or

$$\frac{[(2d_i + \lambda)(k_i + m_i) - d_i \lambda]^2}{(k_i + m_i - \lambda)(d_i + \lambda)} \leq \left(\frac{2m_i \mu_i}{\sigma_i} \right)^2. \tag{33}$$

For each item i , the ordering rule is as follows:

- Order $Q_i = Q_i^*$ units specified by (30) if (33) is satisfied, otherwise
- Order $Q_i = 0$ units if (33) is not satisfied.

The problem now is to find the smallest non-negative λ such that (33) is satisfied. Gallego and Moon (1993) provide a simple line search algorithm to find the optimal value of λ without shortage cost, i.e., $k_i = 0$. Unfortunately, that algorithm may not be valid for the case when shortage cost is considered, since some individual item order quantities that satisfy the budget constraint can be associated with negative expected profit. The following procedure is proposed to handle the case when shortage cost is considered. This new procedure satisfies budget constraints, while eliminating the possibility of negative expected profits for any individual item.

Step 0: Let $T = \{1, \dots, N\}$.

Step 1: Let $\lambda = 0$ and solve for Q_i^* , $i \in T$; Q_i^* is obtained from (30) if (33) is satisfied, $Q_i^* = 0$ if not.

- (a) If (28) is satisfied, go to step 4.
- (b) Otherwise continue to step 2.

Step 2: Starting from $\lambda = 0$, increase λ continuously until the first occurrence of either of the following:

- (a) Constraint (28) is satisfied; go to step 4.
- (b) Condition (33) is violated for one item i , $i \in T$; go to step 3.

Table 1
Data for Example 4

Item i	c_i	p_i	s_i	l_i	μ_i	σ_i
1	35.1	50.3	25.0	14.0	900	122
2	25.0	40.0	12.5	8.0	800	200
3	28.0	32.0	15.1	10.0	1200	170
4	4.8	6.1	2.0	1.5	2300	200

Step 3: Let $Q_i^* = 0$, remove i from set T , and go back to step 1.

Step 4: Stop, the solution is optimum.

Example 4. Three new items are added to the product introduced in Example 1. The total budget is \$80,000 for the four items, whose data are given in Table 1.

Step 0: Let $T = \{1, 2, 3, 4\}$.

Step 1: Assuming $\lambda = 0$, we find that (33) is satisfied for all 4 items. Thus, we solve for the 4 order quantities.

$$Q_1(0) = 968, \quad Q_2(0) = 862, \quad Q_3(0) = 1207, \quad Q_4(0) = 2300,$$

$$\sum c_i Q_i = \$100,363 > \$80,000.$$

Since (28) is not satisfied, we go to step 2

Step 2: Performing a simple line search, we increase the optimal value of the Lagrangian multiplier until:

$$\lambda = 0.438.$$

At that point, (33) is not satisfied for item 3, i.e., the expected profit for item 3 becomes negative. Thus, we go to step 3.

Step 3: Let $Q_3 = 0$. Let $T = \{1, 2, 4\}$. Go to step 1.

Step 1: At $\lambda = 0$, (33) is satisfied for the 3 items in T , and the 3 order quantities are:

$$Q_1(0) = 968, \quad Q_2(0) = 862, \quad Q_4(0) = 2300,$$

$$\sum c_i Q_i = \$66,567 < \$80,000.$$

Since (28) is satisfied, we stop. The solution is optimal.

Lower bounds on expected profit:

Individual items 1, 2, 3, 4 = (11585, 8609, 0, 2430),

Total = \$22,624.

6. Numerical experiments

Extensive numerical experiments have been conducted in order to demonstrate the benefits of the extensions presented in the paper. The aim of these experiments is to evaluate the performance of the new models. Specifically, we compared the expected profit resulting from the new model incorporating shortage cost to that obtained by traditional models. A total of 100 problems, whose randomly generated parameters

Table 2
Summary of the results of computational experiments

	Profit if shortage cost is ignored in calculating Q^*	Profit if shortage cost is considered in calculating Q^*	Percent increase in profit
Minimum	557.8	605.5	0.27
Average	2249.5	2286	1.98
Maximum	4724	4747.3	12.33

are uniformly distributed on the intervals shown below, have been adapted from Moon and Silver (2000):

$$\mu \sim U(50, 150), \quad \sigma \sim U(0.1, 0.3) \times \mu,$$

$$c \sim U(30, 50), \quad p \sim U(1.5, 2.0) \times c,$$

$$s \sim U(0.2, 0.5) \times c, \quad l \sim U(0.4, 0.8) \times c.$$

The numerical experiments are specifically designed to evaluate the improvement in profit resulting from using the new models. Therefore, since an explicit profit expression exists only for the single product case, test problems have been constructed and solved only for this case only. For each problem, two values of Q^* have been calculated using (8), first using the randomly generated value of l , and then assuming $l = 0$. Using (6), the order quantities Q^* resulting from the two methods were then used to determine the corresponding lower bounds on expected profits. The effect of including the shortage cost in calculating Q^* on the expected profit is summarized in Table 2. The average and the maximum increase in expected profit is 2% and 12%, respectively. We expect that a greater profit increase is possible if other cost structures are used.

7. Conclusions

In this work, the models given in Gallego and Moon (1993) have been extended to incorporate shortage penalty cost beyond lost profit. New models and results have been developed for the following cases: the single product case, the fixed ordering cost case, the random yield case, and the resource-constrained multi-product case. Computational experiments have been used to evaluate the improvement in expected profit resulting from the new models. Moreover, a new solution procedure has been developed for the multi-product case with a budget constraint. The inclusion of shortage penalty cost is an important and realistic extension, since in real life there is always a cost associated with failure to satisfy customer demands. Future research ideas include extending the distribution-free newsboy problem to include the following considerations:

- Supplier discounting policies.
- Price-dependent demand, and demand-dependent pricing.
- Deterioration of perishable items.
- Multiple items with substitution.
- Multiple locations, multiple suppliers, and multiple periods.

Acknowledgements

The authors would like to express gratitude to King Fahd University of Petroleum and Minerals for supporting this research effort. Thanks are also due to three anonymous referees for useful comments and suggestions.

References

- Gallego, G., 1992. A minimax distribution free procedure for the (Q, r) inventory model. *Operation Research Letters* 11 (1), 55–60.
- Gallego, G., Moon, I., 1993. The distribution free newsboy problem: review and extensions. *Journal of the Operational Research Society* 44 (8), 825–834.
- Johnson, L., Montgomery, D., 1974. *Operations Research in Production Planning, Scheduling, and Inventory Control*. Wiley, New York.
- Kasugai, H., Kasegai, T., 1960. Characteristics of dynamic maximum ordering policy. *Journal of the Operations Research Society of Japan* 3, 11–26.
- Kasugai, H., Kasegai, T., 1961. Note on minimax ordering policy—static and dynamic solutions and a comparison to maximin policy. *Journal of the Operations Research Society of Japan* 3, 155–169.
- Khouja, M., 1999. The single period (news-vendor) problem: Literature review and suggestions for future research. *Omega* 27 (5), 537–553.
- Moon, I., Choi, S., 1995. Distribution free newsboy problem with balking. *Journal of the Operational Research Society* 46 (4), 537–542.
- Moon, I., Choi, S., 1997. Distribution free procedures for make-to-order (MTO), make-in-advance (MIA), and composite policies. *International Journal of Production Economics* 48 (1), 21–28.
- Moon, I., Silver, E.A., 2000. The multi-item newsvendor problem with a budget constraint and fixed ordering costs. *Journal of the Operational Research Society* 51 (5), 602–608.
- Petruzzi, N.C., Dada, M., 1999. Pricing and the newsvendor problem, a review with extensions. *Operations Research* 47 (2), 183–194.
- Scarf, H., 1958. A min–max solution of an inventory problem. In: Arrow, K., Karlin, S., Scarf, H. (Eds.), *Studies in The Mathematical Theory of Inventory and Production*. Stanford University Press, California, pp. 201–209.
- Silver, E.A., Pyke, D.F., Peterson, R.P., 1998. *Inventory Management and Production Planning and Scheduling*, 3rd Edition. Wiley, New York.
- Vairaktarakis, G.L., 2000. Robust multi-item newsboy models with a budget constraint. *International Journal of Production Economics* 66 (3), 213–226.
- Weatherford, L.R., Pfeifer, P.E., 1994. The economic value of using advance booking of orders. *Omega* 22 (1), 105–111.