Serial #: ________ ID ______________ NAME ____

Show all necessary steps for full marks.

Question 1: (5 points) Find the exact value of each real number y if it exists.

(a): $y=\arctan\left(-\sqrt{3}\right)=?$

(b): $y=\cot^{-1}(-1)=?$

(c): $\cos^{-1}(\cos\frac{6\pi}{5})=?$

(d): $\tan^{-1}\left(\tan\frac{4\pi}{3}\right)=?$

(e): $\sin\left(\cot^{-1}(-3)\right)=?$

Solution: (a): $y=\arctan\left(-\sqrt{3}\right)$

Check that $-\sqrt{3}\in D_{\arctan}=\left(-\infty,\infty\right)$ and $y\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)=$ range of arctan

$y=\arctan\left(-\sqrt{3}\right)\ \Rightarrow\ \tan y=-\sqrt{3}\ \Rightarrow\ \boxed{y=-\frac{\pi}{3}}$

(b): $y=\cot^{-1}(-1)$

Check that $-1\in(-\infty,\infty)=$ domain of $\cot^{-1}$ and $y\in\left(0,\pi\right)=$ Range of $\cot^{-1}$

$y=\cot^{-1}(-1)\ \Rightarrow\ \cot y=\cot\left(\cot^{-1}(-1)\right)=-1\ \Rightarrow\ \cot y=-1\ \Rightarrow\ \boxed{y=\frac{3\pi}{4}}$

(c): $\cos^{-1}(\cos\frac{6\pi}{5})=\cos^{-1}\left(-\cos\left(\frac{6\pi}{5}-\pi\right)\right)=\cos^{-1}\left(-\cos\left(\frac{\pi}{5}\right)\right)=\cos^{-1}(\cos\frac{4\pi}{5})=\frac{4\pi}{5}$

because $\frac{4\pi}{5}\in[0,\pi]=$ restricted domain of cosine

(d): $\tan^{-1}\left(\tan\frac{4\pi}{3}\right)=\tan^{-1}\left(\tan\frac{\pi}{3}\right)=\tan^{-1}\left(\tan\frac{\pi}{3}\right)=\frac{\pi}{3}$

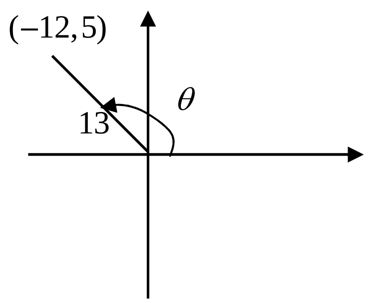
(e): Let $\theta=\cot^{-1}(-3)$. Then $\cot\theta=-3$, where $0<\theta<\pi$.

$\Rightarrow\ \theta\in QII$, $\cot\theta=-3=\frac{-3}{1}\ \Rightarrow\ x=-3, y=1, r=\sqrt{10}$

$\Rightarrow\ \sin\left(\cot^{-1}(-3)\right)=\sin\theta=\frac{y}{r}=\frac{1}{\sqrt{10}}=\frac{\sqrt{10}}{10}$

Question 2: (5 points): Find the exact value of

(a): $\cos\left[\frac{3\pi}{4}-\cos^{-1}\left(-\frac{12}{13}\right)\right]$ **(b):** $\tan\left[2\sin^{-1}\left(-\frac{4}{5}\right)\right]$



Solution

(a): Let $\theta=\cos^{-1}\left(-\frac{12}{13}\right)$. Then $\cos\theta=-\frac{12}{13}$, $0\le\theta\le\pi$

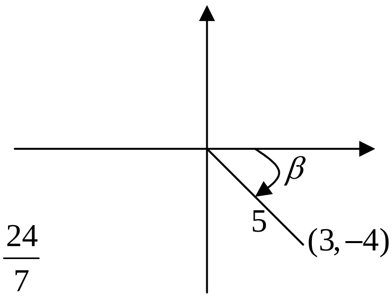
$\Rightarrow\ \theta\in Quadrant\ II$

$\cos\left[\frac{3\pi}{4}-\cos^{-1}\left(-\frac{12}{13}\right)\right]=\cos\left[\frac{3\pi}{4}-\theta\right]$

$=\cos\frac{3\pi}{4}\cos\theta+\sin\frac{3\pi}{4}\sin\theta=-\frac{\sqrt{2}}{2}\cdot\frac{-12}{13}+\frac{\sqrt{2}}{2}\cdot\frac{5}{13}=\frac{12\sqrt{2}}{26}+\frac{5\sqrt{2}}{26}=\frac{17\sqrt{2}}{26}$

(b): Let $\beta = \sin^{-1}\left(-\frac{4}{5}\right)$. Then

$$\sin\beta = -\frac{4}{5}\,,\quad -\frac{\pi}{2} \le \beta \le \frac{\pi}{2} \quad \Rightarrow \quad \theta \in Quadrant\ \ IV$$



$$\tan\left[2\sin^{-1}\left(-\frac{4}{5}\right)\right] = \tan[2\beta] = \frac{2\tan\beta}{1-\tan^2\beta} = \frac{2\left(\frac{-4}{3}\right)}{1-\left(\frac{-4}{3}\right)^2} = \frac{-24}{9-16} = \frac{24}{7}$$

Question 3: (5 points) Verify the identity $\frac{\sec x - 1}{\sec x + 1} - \frac{\sec x + 1}{\sec x - 1} = -4\csc x\cot x$

Solution: $\text{LHS} = \frac{\sec x - 1}{\sec x + 1} - \frac{\sec x + 1}{\sec x - 1} = \frac{(\sec x - 1)^2 - (\sec x + 1)^2}{(\sec x + 1)(\sec x - 1)} = \frac{\sec^2 x - 2\sec x + 1 - (\sec^2 x + 2\sec x + 1)}{\sec^2 x - 1}$

$$= \frac{-4\sec x}{\tan^2 x} = -4\frac{\frac{1}{\cos x}}{\frac{\sin^2 x}{\cos^2 x}} = -4\frac{\cos x}{\sin^2 x} = -4\frac{1}{\sin x}\frac{\cos x}{\sin x} = -4\csc x\cot x = \text{RHS}$$

Question 4: (5 points) (7.2Textbook Example 6): Write the expression $\sin(\cos^{-1}x + \tan^{-1}y)$ in terms of x and y where $-1 \le x \le 1$ and y is any real number.

Solution:

See example 6 page 584.

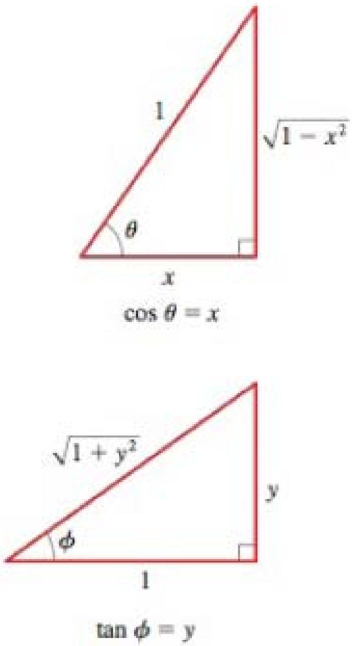


FIGURE 2

EXAMPLE 6 ■ **Simplifying an Expression Involving Inverse Trigonometric Functions**

Write $\sin(\cos^{-1}x + \tan^{-1}y)$ as an algebraic expression in x and y, where $-1 \le x \le 1$ and y is any real number.

SOLUTION Let $\theta = \cos^{-1}x$ and $\phi = \tan^{-1}y$. Using the methods of Section 5.4, we sketch triangles with angles θ and ϕ such that $\cos\theta = x$ and $\tan\phi = y$ (see Figure 2). From the triangles we have

$$\sin\theta = \sqrt{1-x^2} \qquad \cos\phi = \frac{1}{\sqrt{1+y^2}} \qquad \sin\phi = \frac{y}{\sqrt{1+y^2}}$$

From the Addition Formula for Sine we have

$$\sin(\cos^{-1}x + \tan^{-1}y) = \sin(\theta + \phi)$$

$$= \sin\theta\cos\phi + \cos\theta\sin\phi \qquad \text{Addition Formula for Sine}$$

$$= \sqrt{1-x^2}\,\frac{1}{\sqrt{1+y^2}} + x\frac{y}{\sqrt{1+y^2}} \qquad \text{From triangles}$$

$$= \frac{1}{\sqrt{1+y^2}}\left(\sqrt{1-x^2} + xy\right) \qquad \text{Factor } \frac{1}{\sqrt{1+y^2}}$$