REDUCTION IDENTITY:

$a\sin x + b\cos x = \sqrt{a^2+b^2}\sin(x+\alpha)$

where a and b are nonzero real numbers. α is determined by the equations

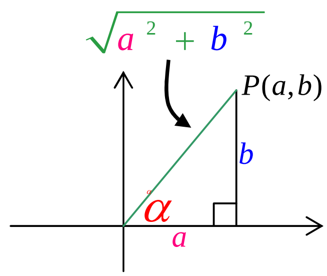
$\cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$ and $\sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$.

Note that a is the coefficient of sine and b is the coefficient of cosine.

The function given by $f(x) = a\sin x + b\cos x$ can be written in the form $f(x) = \sqrt{a^2+b^2}\sin(x+\alpha)$. This form of the function is useful in graphing and engineering applications because the amplitude, period, and phase shift can be readily calculated.

Let $P(a,b)$ be a point on a coordinate plane, and let α represent an angle in standard position

We show that $a\sin x + b\cos x = \sqrt{a^2+b^2}\sin(x+\alpha)$:



$$a\sin x + b\cos x = \frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}}(a\sin x + b\cos x)$$

$$= \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x\right)$$

$$= \sqrt{a^2+b^2}\left(\cos\alpha \sin x + \sin\alpha \cos x\right) \quad \text{by the Figure}$$

$$= \sqrt{a^2+b^2}\left(\sin x \cos\alpha + \cos x \sin\alpha\right)$$

$$= \sqrt{a^2+b^2}\sin(x+\alpha), \text{ where } \cos\alpha = \frac{a}{\sqrt{a^2+b^2}}, \ \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}.$$

Exercise 1: Rewrite the function $f(x) = 3\sin x - \sqrt{3}\cos x$ as a single sine function.

Solution: $f(x) = 3\sin x - \sqrt{3}\cos x \Rightarrow a = 3, \ b = -\sqrt{3}$

$$k = \sqrt{a^2+b^2} = \sqrt{(3)^2 + \left(-\sqrt{3}\right)^2} = \sqrt{9+3} = \sqrt{12} = 2\sqrt{3}$$

$$\left.\begin{array}{l} \sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}} = \dfrac{-\sqrt{3}}{2\sqrt{3}} = -\dfrac{1}{2} \\ \cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}} = \dfrac{3}{2\sqrt{3}} = \dfrac{3}{2\sqrt{3}}\cdot\dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}}{2} \end{array}\right\} \Rightarrow \begin{cases} \alpha \in \text{IV} \\ \alpha = 330^\circ = \dfrac{11\pi}{6} \text{ or } \alpha = -30^\circ = -\dfrac{\pi}{6} \end{cases}$$

$$f(x) = 3\sin x - \sqrt{3}\cos x = \sqrt{a^2+b^2}\sin(x+\alpha) = 2\sqrt{3}\sin\left(x + \frac{11\pi}{6}\right) = 2\sqrt{3}\sin\left(x + 330^\circ\right)$$

Or

$$f(x) = 3\sin x - \sqrt{3}\cos x = \sqrt{a^2+b^2}\sin(x+\alpha) = 2\sqrt{3}\sin\left(x - \frac{\pi}{6}\right) = 2\sqrt{3}\sin\left(x - 30^\circ\right)$$

Exercise 2: Given the function $f(x)=2\sin\frac{x}{3}-2\sqrt{3}\cos\frac{x}{3}$

(a): Rewrite $f(x)$ in the form $f(x)=k\sin(bx+\alpha)$

(b): Find the amplitude, the phase shift, the period, and the range for the graph of $f(x)$.

(c): Sketch the graph of the function $f(x)=2\sin\frac{x}{3}-2\sqrt{3}\cos\frac{x}{3}$ over two periods.

Solution: (a): $f(x)=a\sin\frac{x}{3}+b\cos\frac{x}{3}=k\sin\left(\frac{x}{3}+\alpha\right)$

$a=2,\ b=-2\sqrt{3}\ \Rightarrow\ (2,-2\sqrt{3})$ is in Quadrant IV.

$k=\sqrt{a^2+b^2}=\sqrt{2^2+(-2\sqrt{3})^2}=\sqrt{4+12}=4$

$$\left.\begin{array}{l}\sin\alpha=\frac{b}{k}=\frac{-2\sqrt{3}}{4}=-\frac{\sqrt{3}}{2}\\ \cos\alpha=\frac{a}{k}=\frac{2}{4}=\frac{1}{2}\end{array}\right\}\Rightarrow\ \alpha \text{ is in Quadrant IV and } \alpha=-\frac{\pi}{3}\ \text{ OR }\ \alpha=\frac{5\pi}{3}$$

$f(x)=4\sin\left(\frac{x}{3}-\frac{\pi}{3}\right)$ OR $f(x)=4\sin\left(\frac{x}{3}+\frac{5\pi}{3}\right)$

(b): Amplitude $=4$ Phase shift $=-\frac{-\frac{\pi}{3}}{\frac{1}{3}}=\pi$ units to the right.

OR Phase shift $=-\frac{5\pi/3}{1/3}=-5\pi$ $|-5\pi|$ units to the left.

Period $=\frac{2\pi}{1/3}=6\pi$ Range $=[-4,4]$

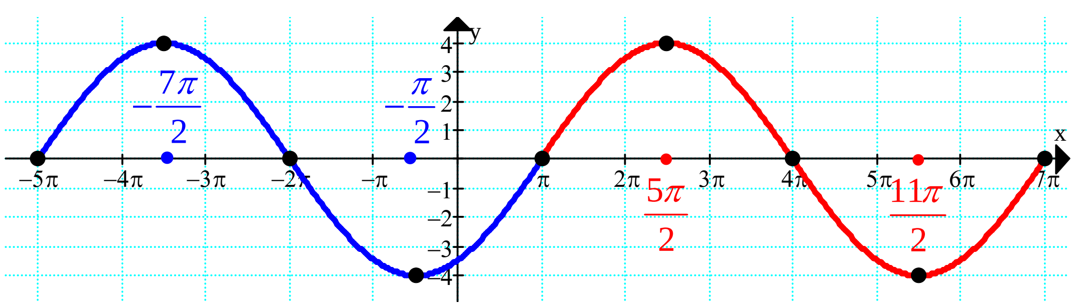
(c): Graph of $f(x)=4\sin\left(\frac{x}{3}+\frac{5\pi}{3}\right)$ is: Graph of $f(x)=4\sin\left(\frac{x}{3}-\frac{\pi}{3}\right)$ is:

Beginning key point $=$ Phase shift $=\pi$ **OR** beginning key point $=$ Pha shift $=-5\pi$

First quarter key point $=$ phase shift $+\frac{1}{4}$period $=\pi+\frac{1}{4}(6\pi)=\pi+\frac{3\pi}{2}=\frac{5\pi}{2}$

The key points are $\pi=\frac{2\pi}{2}$, $\frac{5\pi}{2}$, $\frac{8\pi}{2}=4\pi$, $\frac{11\pi}{2}$, $\frac{14\pi}{2}=7\pi$

The key points are -5π , $-5\pi+\frac{1}{4}Period=-5\pi+\frac{3\pi}{2}=-\frac{7\pi}{2}$, $-\frac{4\pi}{2}$, $-\frac{\pi}{2}$



Additional Exercises:

Exercise 3: Rewrite the function $f(x) = -3\sin 2x + \sqrt{3}\cos 2x$ as a single sine function.

Solution: $f(x) = -3\sin 2x + \sqrt{3}\cos 2x \Rightarrow a = -3 \ , \ b = \sqrt{3}$

$$k = \sqrt{a^2+b^2} = \sqrt{(-3)^2 + (\sqrt{3})^2} = \sqrt{9+3} = \sqrt{12} = 2\sqrt{3}$$

$$\left.\begin{array}{l} \sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}} = \dfrac{\sqrt{3}}{2\sqrt{3}} = \dfrac{1}{2} \\ \cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}} = \dfrac{-3}{2\sqrt{3}} = \dfrac{-3}{2\sqrt{3}}\cdot\dfrac{\sqrt{3}}{\sqrt{3}} = -\dfrac{\sqrt{3}}{2} \end{array}\right\} \Rightarrow \begin{cases} \alpha \in \text{II} \\ \alpha = 150^\circ = \dfrac{5\pi}{6} \text{ or } \alpha = -210^\circ = -\dfrac{7\pi}{6} \end{cases}$$

$$f(x) = -3\sin 2x + \sqrt{3}\cos 2x = \sqrt{a^2+b^2}\sin(2x+\alpha) = 2\sqrt{3}\sin\left(2x + \frac{5\pi}{6}\right) = 2\sqrt{3}\sin(2x + 150^\circ)$$

or $f(x) = -3\sin 2x + \sqrt{3}\cos 2x = \sqrt{a^2+b^2}\sin(2x+\alpha) = 2\sqrt{3}\sin\left(2x - \frac{7\pi}{6}\right) = 2\sqrt{3}\sin(2x - 210^\circ)$

Exercise 4: If $\sin x + \cos x$ is written as $k\sin(x+\alpha)$ then find k and the measure of α.

Solution: Here $a = 1 \ , \ b = 1$

$$\sqrt{a^2+b^2} = \sqrt{(1)^2+(1)^2} = \sqrt{1+1} = \sqrt{2}$$

$$\left.\begin{array}{l} \sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2} \\ \cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2} \end{array}\right\} \Rightarrow \begin{cases} \alpha \in \text{I} \\ \alpha = 45^\circ = \dfrac{\pi}{4} \text{ or } \alpha = -315^\circ = -\dfrac{7\pi}{4} \end{cases}$$

$$\sin x + \cos x = \sqrt{a^2+b^2}\sin(x+\alpha) = \sqrt{2}\sin(x+45^\circ) = \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$$

$$\sqrt{2}\sin(x+45^\circ-360^\circ) = \sqrt{2}\sin(x-315^\circ)$$

$$\sqrt{2}\sin\left(x+\frac{\pi}{4}\right) = \sqrt{2}\sin\left(x+\frac{\pi}{4}-2\pi\right) = \sqrt{2}\sin\left(x-\frac{7\pi}{4}\right)$$

Exercise 5: Given $f(x) = -\frac{1}{2}\cos 2x + \frac{\sqrt{3}}{2}\sin 2x - 4$. Find the following

(a): the range of f **(b):** the period of f **(c):** the amplitude of f

(d): Write $f(x) = -\frac{1}{2}\cos 2x + \frac{\sqrt{3}}{2}\sin 2x - 4$ in the form $y = k\sin(2x+\alpha) - 4$ where the measure of α is in radian and $k = \sqrt{a^2+b^2}$.

(e): the phase shift of f.

(f): Sketch the graph of f over $\left[-\frac{11\pi}{12}, \frac{13\pi}{12}\right]$

(g): Sketch the graph of $g(x) = f(x) = \csc(2x - \frac{\pi}{6}) - 4$ over $\left[-\frac{11\pi}{12}, \frac{13\pi}{12}\right]$

Solution: $a=\frac{\sqrt{3}}{2},\ b=-\frac{1}{2}\ \Rightarrow\ \alpha$ is in Quadrant IV because $(a,b)=\left(\sqrt{3}/2,-1/2\right)$ is in Quadrant IV.

$$k=\sqrt{a^2+b^2}=\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2+\left(-\frac{1}{2}\right)^2}=\sqrt{\frac{3}{4}+\frac{1}{4}}=1$$

$$\left.\begin{aligned}\sin\alpha&=\frac{b}{k}=-\frac{1}{2}\\ \cos\alpha&=\frac{a}{k}=\frac{\sqrt{3}}{2}\end{aligned}\right\}\Rightarrow\ \begin{aligned}&\alpha\in IV\\ &\alpha=-\frac{\pi}{6}\ \text{ OR }\ \alpha=-\frac{\pi}{6}+2\pi=\frac{11\pi}{6}\end{aligned}$$

$$f(x)=\frac{\sqrt{3}}{2}\sin 2x-\frac{1}{2}\cos 2x-4=k\sin(2x+\alpha)-4=\sin(2x-\frac{\pi}{6})-4$$

OR $f(x)=\frac{\sqrt{3}}{2}\sin 2x-\frac{1}{2}\cos 2x-4=k\sin(2x+\alpha)-4=\sin(2x+\frac{11\pi}{6})-4$

(a): $R_f=[-5,-3]$ **(b):** $P=\frac{2\pi}{2}=\pi$ **(c):** Amp $=1$

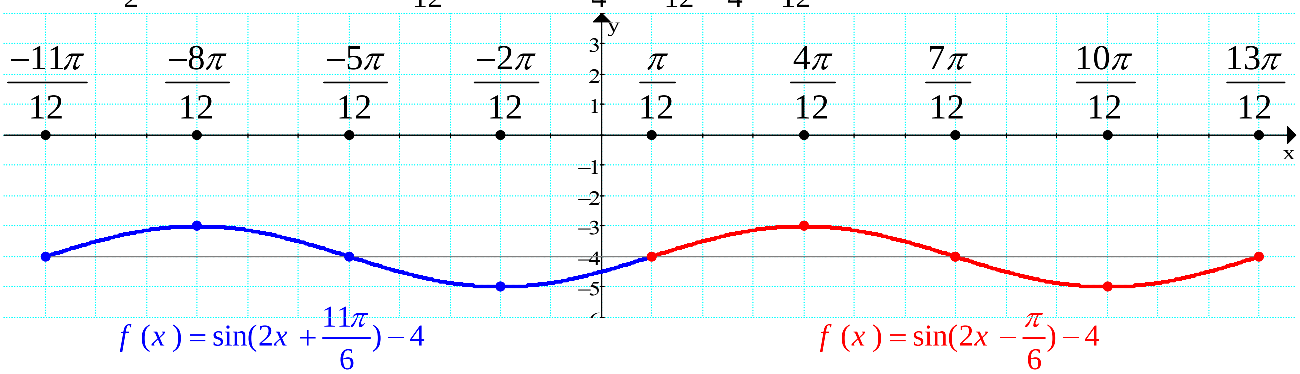
(d): $f(x)=\sin(2x-\frac{\pi}{6})-4,\ f(x)=\sin(2x+\frac{11\pi}{6})-4$

(e): The phase shift of $f(x)=\frac{\sqrt{3}}{2}\sin 2x-\frac{1}{2}\cos 2x-4=\sin(2x-\frac{\pi}{6})-4$ is: $\frac{\pi}{12}$, ($\frac{\pi}{12}$ to the right)

The phase shift of $f(x)=\frac{\sqrt{3}}{2}\sin 2x-\frac{1}{2}\cos 2x-4=\sin(2x+\frac{11\pi}{6})-4$ is: $-\frac{11\pi}{12}$, ($\frac{11\pi}{12}$ to the left)

(f): $f(x)=\sin(2x-\frac{\pi}{6})-4$

$\Rightarrow\ P=\frac{2\pi}{2}=\pi,\quad PhaseShift=\frac{\pi}{12}\ \Rightarrow\ PS+\frac{1}{4}P=\frac{\pi}{12}+\frac{\pi}{4}=\frac{4\pi}{12}$



Exercise 6:

If $\sin 20^\circ-\sqrt{3}\cos 20^\circ=k\sin\theta,\ 0^\circ<\theta<90^\circ$, then k and θ are equal to

A) $-2,\ 40^\circ$

B) $2,\ 20^\circ$

C) $1-\sqrt{3},\ 20^\circ$

D) $-2,\ 20^\circ$

E) $-2,\ 30^\circ$

Solution: $\sin 20^\circ - \sqrt{3}\cos 20^\circ = \sqrt{a^2+b^2}\,\sin\left(20^\circ+\alpha\right) = k\,\sin\theta,\ \ 0^\circ < \theta < 90^\circ$

$$\sqrt{a^2+b^2} = \sqrt{1^2+\left(-\sqrt{3}\right)^2} = 2$$

$$\left.\begin{array}{l} \sin\alpha = \dfrac{b}{k} = \dfrac{-\sqrt{3}}{2} \\ \cos\alpha = \dfrac{a}{k} = \dfrac{1}{2} \end{array}\right\} \Rightarrow \quad \begin{array}{l} \alpha \in \text{IV} \\ \boxed{\alpha = -60^\circ} \text{ OR } \boxed{\alpha = 300^\circ} \end{array}$$

$\alpha = -60^\circ$: $\sin 20^\circ - \sqrt{3}\cos 20^\circ = k\,\sin\left(20^\circ+\alpha\right)$

$$= 2\sin\left(20^\circ - 60^\circ\right)$$
$$= 2\sin\left(-40^\circ\right)$$
$$= -2\sin 40^\circ = k\,\sin\theta \quad \Rightarrow \quad \boxed{k = -2},\ \boxed{\theta = 40^\circ}$$

OR $\alpha = 300^\circ$: $\sin 20^\circ - \sqrt{3}\cos 20^\circ = k\,\sin\left(20^\circ+\alpha\right)$

$$= 2\sin\left(20^\circ + 300^\circ\right)$$
$$= 2\sin\left(320^\circ\right)$$
$$= -2\sin\left(-320^\circ\right)$$
$$= -2\sin\left(-320^\circ + 360^\circ\right)$$
$$= -2\sin\left(40^\circ\right)$$

Exercise 7: The range of $f(x) = \sqrt{3}\sin x - \cos x + 2$ is

(a): $[0,4]$

(b): $[-2,2]$

(c): $[1,3]$

(d): $[-1,1]$

(e): $\left[-1,\sqrt{3}\right]$

Solution: $a = \sqrt{3}$, $b = -1 \ \Rightarrow\ k = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$

$$f(x) = \sqrt{3}\sin x - \cos x + 2 = k\,\sin\left(x+\alpha\right) + 2 = 2\sin\left(x+\alpha\right) + 2$$

$$Range = \left[-|a| + d\ ,\ |a| + d\right]$$
$$= \left[-2+2\ ,\ 2+2\right]$$
$$= \left[0\ ,\ 4\right]$$

Exercise 8: If $-\sqrt{3}\sin x + \cos x = A\sin(x+\theta)$, where $0^\circ \le \theta \le 180^\circ$, then find the value of A and θ.

Answer: $A = \sqrt{(-\sqrt{3})^2 + (1)^2} = 2 \qquad \theta = 150^\circ = \dfrac{5\pi}{6}$

Exercise 9: Find the range of the function $f(x)=\dfrac{10}{-3\sin x+4\cos x}-4$

Answer: $Range=(-\infty,-6]\cup[-2,\infty)$

Exercise 10: If $\sin 40°+\cos 40°=k\sin(\beta)$. Find the values of k and β.

Answer: $\boxed{k=\sqrt{2}}$ $\boxed{\beta=85°}$

Exercise 11: Find the amplitude and range of $y=3\sin\dfrac{x}{2}+4\cos\dfrac{x}{2}$. **Answer:** $\text{Amp}=5$, $\text{Range}=[-5,5]$

Exercise 12: If the function $f(x)=-\sin 2x+\sqrt{3}\cos 2x$ is written in the form $f(x)=k\sin(bx+\alpha)$ the phase shift of $f(x)$ is

A) $-\dfrac{\pi}{6}$ B) $\dfrac{2\pi}{3}$ C) $\dfrac{\pi}{6}$ D) $-\dfrac{\pi}{3}$ E) $\dfrac{\pi}{3}$

Exercise 13: The expression $-\sqrt{2}\sin\dfrac{\pi}{5}+\sqrt{2}\cos\dfrac{\pi}{5}$ can be written as

A) $2\sin\dfrac{\pi}{20}$ B) $2\sin\dfrac{9\pi}{20}$ C) $-2\sin\dfrac{9\pi}{20}$ D) $-2\cos\dfrac{9\pi}{20}$ B) $2\sin\dfrac{7\pi}{20}$

Solution: $-\sqrt{2}\sin\dfrac{\pi}{5}+\sqrt{2}\cos\dfrac{\pi}{5}=k\sin\left(\dfrac{\pi}{5}+\alpha\right)$

$$k=\sqrt{\left(-\sqrt{2}\right)^2+\left(\sqrt{2}\right)^2}=2$$

$$\left.\begin{aligned}\sin\alpha&=\frac{\sqrt{2}}{2}\\ \cos\alpha&=-\frac{\sqrt{2}}{2}\end{aligned}\right\}\Rightarrow \alpha=\frac{3\pi}{4}$$

$$-\sqrt{2}\sin\frac{\pi}{5}+\sqrt{2}\cos\frac{\pi}{5}=k\sin\left(\frac{\pi}{5}+\alpha\right)=2\sin\left(\frac{\pi}{5}+\frac{3\pi}{4}\right)=2\sin\left(\frac{19\pi}{20}\right)$$

$$=2\sin\left(\pi-\frac{19\pi}{20}\right)=2\sin\left(\frac{\pi}{20}\right)$$

Exercise 14: The minimum value of $f(x)=5\sqrt{2}\sin\left(\dfrac{x}{2}\right)-5\sqrt{2}\cos\left(\dfrac{x}{2}\right)-2$ is

A) -12 B) -8 C) -10 D) -2 E) -3

Solution:

$$y=5\sqrt{2}\sin\left(\frac{x}{2}\right)-5\sqrt{2}\cos\left(\frac{x}{2}\right)-2=\sqrt{\left(5\sqrt{2}\right)^2+\left(-5\sqrt{2}\right)^2}\sin\left(\frac{x}{2}+\alpha\right)=10\sin\left(\frac{x}{2}+\alpha\right)$$

$$\begin{aligned} Range &= \left[-|a|+d\ ,|a|+d\ \right] \\ &= [-10-2\,,10-2] \\ &= [-12\,,8] \end{aligned}$$

$$\text{minimum}=-12$$

Exercise 15: The range of the function $f(x)=\frac{-3}{\csc x}+\frac{4}{\sec x}+3$ is

A) $[0,4]$ B) $[-2,8]$ C) $[-2,4]$ D) $[-5,5]$ E) $[-4,4]$

Solution: $f(x)=\frac{-3}{\csc x}+\frac{4}{\sec x}+3=-3\sin x+4\cos x+3$

$$f(x)=\sqrt{(-3)^2+4^2}\sin(x+\alpha)+3=5\sin(x+\alpha)+3$$

$$Range=\left[-|a|+d\,,|a|+d\ \right]=[-5+3,5+3]=[-2,8]$$