4.1 INVERSE FUNCTIONS

Inverse Function

If the coordinates of the ordered pairs of a function *g* are the reverse of the coordinates of the ordered pairs of a function *f*, then *g* is said to be the *inverse function* of *f*.

Ex1:

Given the function $f = \{(1, -1), (2, 3), (-3, 1), (5, 2)\}$

- The inverse relation of f is $g=\{(-1, 1), (3, 2), (1, -3), (2, 5)\}$.
- The domain of f is {1, 2, -3, 5} and the range is {-1, 3,1,2}.
- The domain of g is {-1, 3,1,2} and the range is {1, 2, -3, 5}.

Notes

- The *domain* of the inverse relation is the *range* of the original function.
- The *range* of the inverse relation is the *domain* of the original function.
- If *g* is the inverse of f then f is also the inverse of *g*.
- We denote to the inverse function of f *by* f^{-1}
- $(f^{-1})^{-1} = f$

Condition for the Inverse Function

A function f has an **inverse function** if and only if f is a **one-to-one**.

Alternative Condition for an Inverse Function

If f is an increasing function or a decreasing function, then f has an inverse function.

Ex2:

Which of the following functions has an inverse function?

a) $f(x)=x^3$

b) $g(x)=x^2$

c) $h(x) = x^2, \quad x \geq 0$

Restricted domain

Solution

a) f is 1-1, thus f has inverse.

b) g is not 1-1, thus g has no inverse.

c) h is 1-1, thus h has inverse.

Composition of Inverse Functions Property

If f is a one-to-one function, then f^{-1} is the inverse function of f if and only if

$(f \circ f^{-1})(x) = f[f^{-1}(x)] = x$ for all x in the domain of f^{-1}

and

$(f^{-1} \circ f)(x) = f^{-1}[f(x)] = x$ for all x in the domain of f.

Ex3:

Use the composition of functions to show $f^{-1}(x) = \frac{1}{4}x + 8$ is the inverse of $f(x) = 4x - 32.$

Solution

We must show that

$(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(\frac{1}{4}x + 8) = 4(\frac{1}{4}x + 8) - 32 = x + 32 - 32 = x$$

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(4x - 32) = \frac{1}{4}(4x - 32) + 8 = x - 8 + 8 = x$$

How to Find The Inverse of a Function?

To find the inverse function of one-to-one function interchange x and y and solve for y.

Ex4: Find the inverse function of $f(x) = 3x + 5$.

Solution

$y = 3x + 5$ Original equation

$x = 3y + 5$ Switch x and y

$3y + 5 = x$ Reverse sides of the equation.

$y = \dfrac{x-5}{3}$ Solve for y

Thus $f^{-1}(x) = \dfrac{x-5}{3}$

Ex5:

Find the inverse of the following functions. State the domain and the range of the inverse function.

$$a)\quad f(x)=\frac{2x-1}{x+3},\, x\neq -3$$

$$b)\quad f(x)=x^2+4x+3,\quad x\geq -2$$

$$c)\quad f(x)=x^2-6x,\quad x\leq 3$$

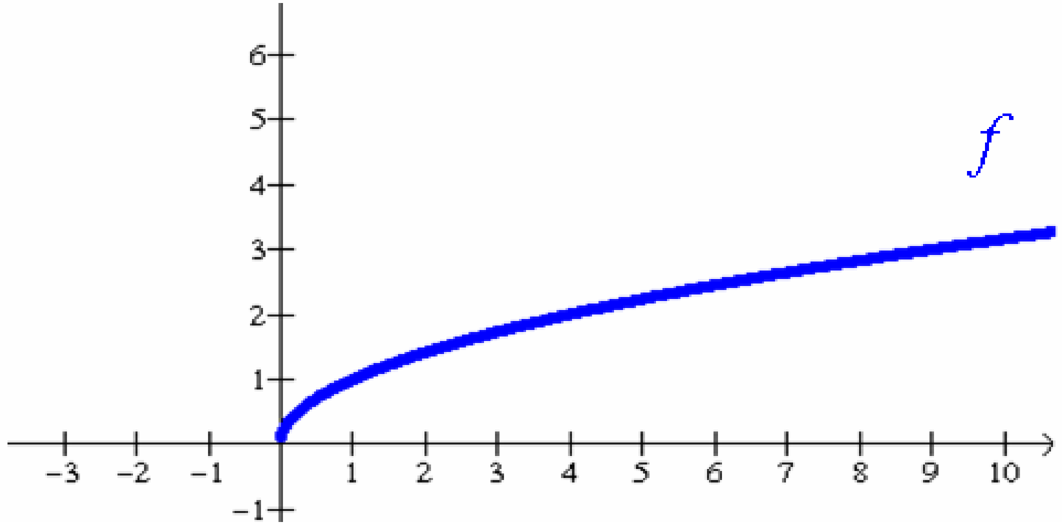
$$d)\quad g(x)=\sqrt{4-x},\quad x\leq 4$$

Graphs of Inverse Function

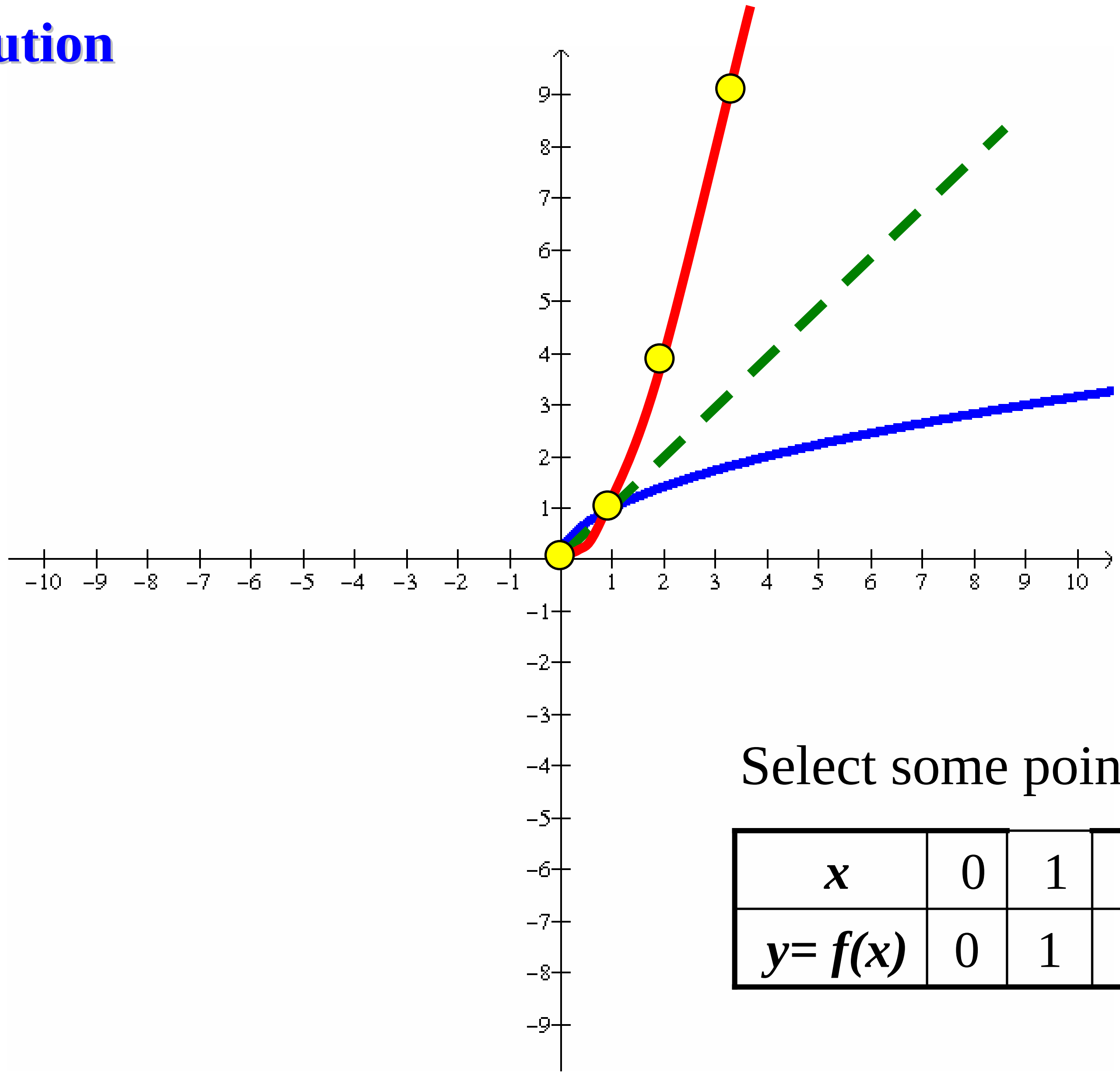
To sketch the graph of f^{-1}, we can use the following properties:

- The graph of f contains the point (a, b) if and only if f^{-1} contains the point (b, a)
- The graph of f^{-1} is symmetric to the graph of f with respect to the line $y = x$

Ex6: Use the graph of f to sketch the graph of f^{-1}



Solution



Select some points on f

x	0	1	4	9
$y = f(x)$	0	1	2	3

Ex7:

Let $f(x) = -x^2-3x+k$ such that f-1(x) exists. If $f^{-1}(2) = 3$, then find k

Ex8:

$$\text{If } f(x) = \frac{2x+1}{x-1}, \text{ then find } (f \circ f^{-1})(5) + f^{-1}(1)$$

Ex9:

$$\text{If } f^{-1}(x) = 2 + \sqrt{x-1}, x \geq 1 \text{ then find } f(4)$$