3.5 Graphing Rational Functions

Def.:

A **rational function** is a function of the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$.

Example: $f(x) = \frac{1}{x}$ is defined for all real numbers except $x = 0$.

x	*f(x)*
2	0.5
1	1
0.5	2
0.1	10
0.01	100
0.001	1000

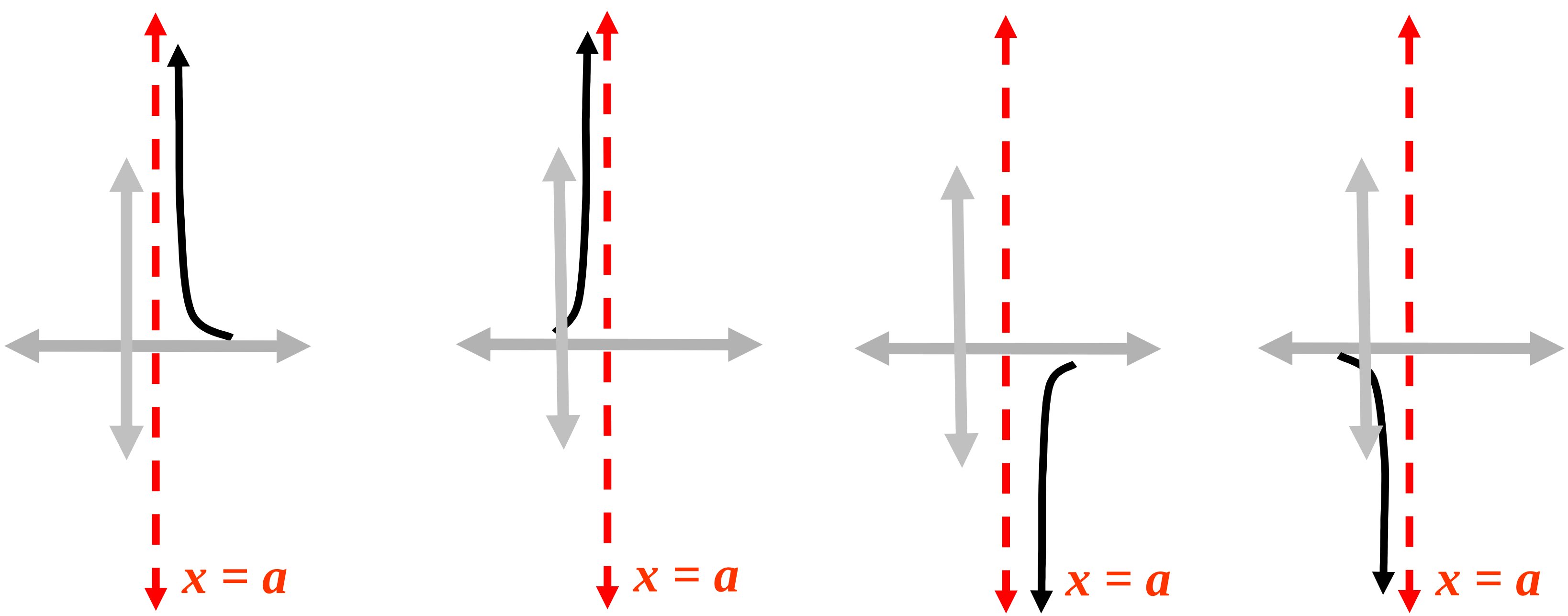
$f(x) \to +\infty$ as $x \to 0^+$

x	*f(x)*
-2	-0.5
-1	-1
-0.5	-2
-0.1	-10
-0.01	-100
-0.001	-1000

$f(x) \to -\infty$ as $x \to 0^-$

Def.:

The line $x = a$ is a **vertical asymptote** (**خط تقارب عمودي**) of the graph of f Provided $f(x) \to +\infty$ or $f(x) \to -\infty$ as $x \to a$ from either left or right.



$f(x) \to +\infty$ as $x \to a^+$

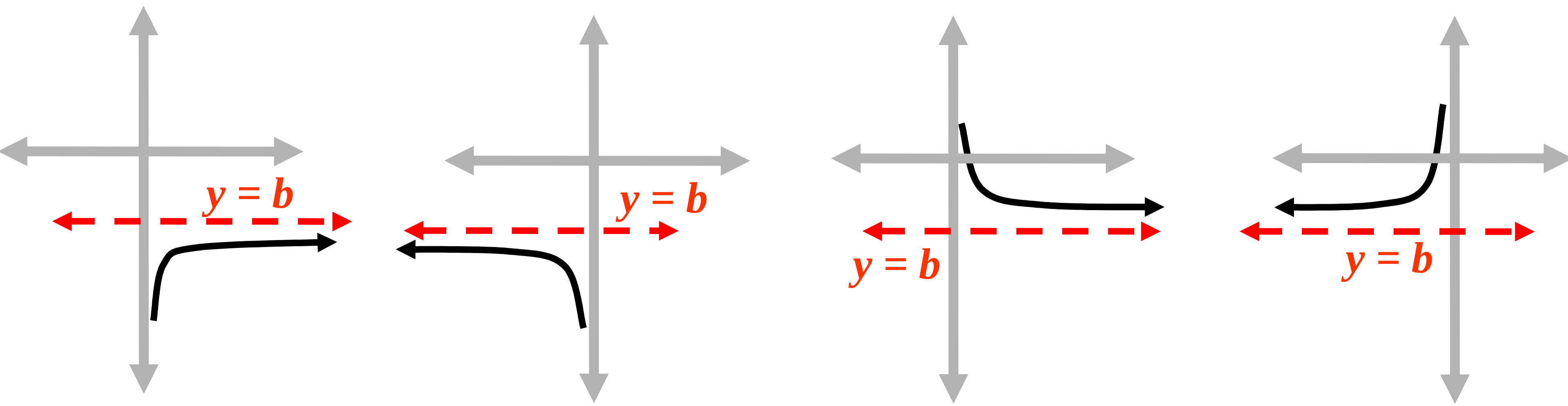
$f(x) \to +\infty$ as $x \to a^-$

$f(x) \to -\infty$ as $x \to a^+$

$f(x) \to -\infty$ as $x \to a^-$

Def.:

The line $y = b$ is a **horizontal asymptote** (خط تقارب أفقي) of the graph of f provided $f(x) \to b$ as $x \to +\infty$ or as $x \to -\infty$.



$f(x) \to b$ as $x \to +\infty$	$f(x) \to b$ as $x \to -\infty$	$f(x) \to b$ as $x \to +\infty$	$f(x) \to b$ as $x \to -\infty$

Theorem on Vertical Asymptotes

If the real number a is a zero of the denominator $Q(x)$, then the graph of $F(x) = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ have no common factors, has the vertical asymptote $x = a$.

Ex1:

Find the vertical asymptote of each rational function.

$$a)\quad f(x) = \frac{x}{x^2 - 3x - 4}$$

$$b)\quad g(x) = \frac{x-1}{x^2 - 1}$$

$$c)\quad h(x) = \frac{-2x^2}{3x^2 + 1}$$

Theorem on Horizontal Asymptotes

Let $$F(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdot \cdot \cdot + b_1 x + b_0}$$

be a rational function with numerator of degree n and denominator of degree m.

1. If $n < m$, then the x-axis, which is the line given by $y = 0$, is the horizontal asymptote of the graph of F.
2. If $n = m$, then the line given by $y = a_n/b_m$ is the horizontal asymptote of the graph of F.
3. If $n > m$, the graph of F has no horizontal asymptote.

<u>Ex2:</u> Find the vertical asymptote of each rational function.

$a)\ f(x) = \dfrac{2x+1}{x^2-3x-4}$ $\quad b)\ g(x) = \dfrac{3x^2-1}{-2x^2+x-1}$ $\quad c)\ h(x) = \dfrac{-2x^5}{3x^2+1}$

• Theorem in slant asymptotes

The rational function given by $F(x)=P(x)/Q(x)$, where $P(x)$ and $Q(x)$ have no common factors, has *a slant asymptote* if the degree of the polynomial $P(x)$ in the numerator is one greater than the degree of the polynomial $Q(x)$ in the denominator.

• How to find the slant asymptote?

By long division. $F(x)=\dfrac{P(x)}{Q(x)}=(mx+b)+\dfrac{r(x)}{Q(x)},$

$(mx+b)$ ↓ slant asymptote

Ex3:

Find the slant asymptote of $f(x) = \dfrac{4x^3 + 7x^2 + 22x - 8}{x^2 + 2x + 5}$

Solution

$$4x - 1$$

$$x^2 + 2x + 5 \,\big)\, 4x^3 + 7x^2 + 22x - 8$$

$$4x^2 + 8x^2 + 20x$$

$$-x^2 + 2x - 8$$

$$-x^2 - 2x - 5$$

$$4x - 3$$

Therefore the slant asymptote is given by $y = 4x - 1$

Ex4:

Determine all asymptotes for the graph of the following functions.

$$a)\ \ F(x) = \frac{2x^2 - 4x + 5}{x - 3}$$

$$b)\ \ G(x) = \frac{2x(x^2 - 4x - 12)}{x^3 - 36x}$$

$$c)\ \ h(x) = \frac{x^2 - 1}{x^3 - 1}$$

Ex5:

Give an example of a rational function without any asymptote.

Ex6:

If y = -3 is a horizontal asymptote for the graph of $F(x) = \dfrac{ax^2 + x + 2}{2x^2 - x - 1}$,

then find the vertical asymptote(s).

General Procedure for Graphing Rational Functions That Have No Common Factors

1. *Asymptotes* Find the real zeros of the denominator $Q(x)$. For each zero a, draw the dashed line $x = a$. Each line is a vertical asymptote of the graph of F. Also graph any horizontal asymptotes.

2. *Intercepts* Find the real zeros of the numerator $P(x)$. For each real zero c, plot the point $(c, 0)$. Each such point is an x-intercept of the graph of F. For each x-intercept use the even and odd powers of $(x - c)$ to determine if the graph crosses the x-axis at the intercept or if the graph intersects but does not cross the x-axis. Also evaluate $F(0)$. Plot $(0, F(0))$, the y-intercept of the graph of F.

3. *Symmetry* Use the tests for symmetry to determine whether the graph of the function has symmetry with respect to the y-axis or symmetry with respect to the origin.

4. *Additional points* Plot some points that lie in the intervals between and beyond the vertical asymptotes and the x-intercepts.

5. *Behavior near asymptotes* If $x = a$ is a vertical asymptote, determine whether $F(x) \to \infty$ or $F(x) \to -\infty$ as $x \to a^-$ and also as $x \to a^+$.

6. *Complete the sketch* Use all the information obtained above to sketch the graph of F.

Ex7: Sketch the graph of $f(x)=\dfrac{x+2}{x-2}$

Solution

Asymptotes

V.A.: x=2

H.A.: y=1

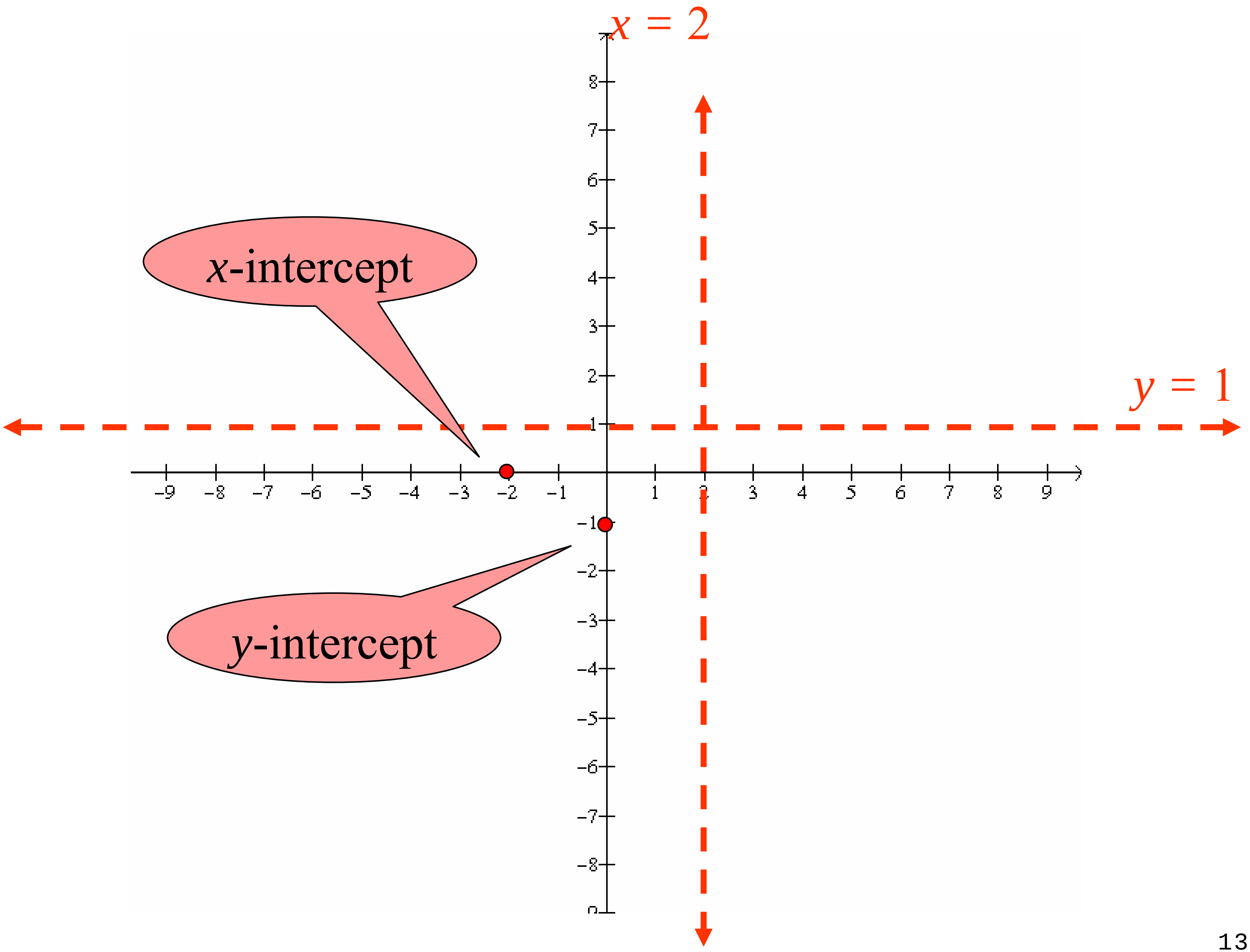
Intercepts

- x-intercept: put $y=0 \Rightarrow x+2=0 \Rightarrow x=-2 \Rightarrow (-2,0)$
- y-intercept: put $x=0 \Rightarrow y=-1 \Rightarrow (0,-1)$

Check if the graph intersects the H.A.

Solve the equation: $f(x)=1 \Rightarrow x+2=x-2 \Rightarrow 2=-2\,!$

Therefore, the graph does not intersect the H.A.


$x = 2$
$y = 1$
x-intercept
y-intercept
8
7
6
5
4
3
2
1
−1
−2
−3
−4
−5
−6
−7
−8
−9
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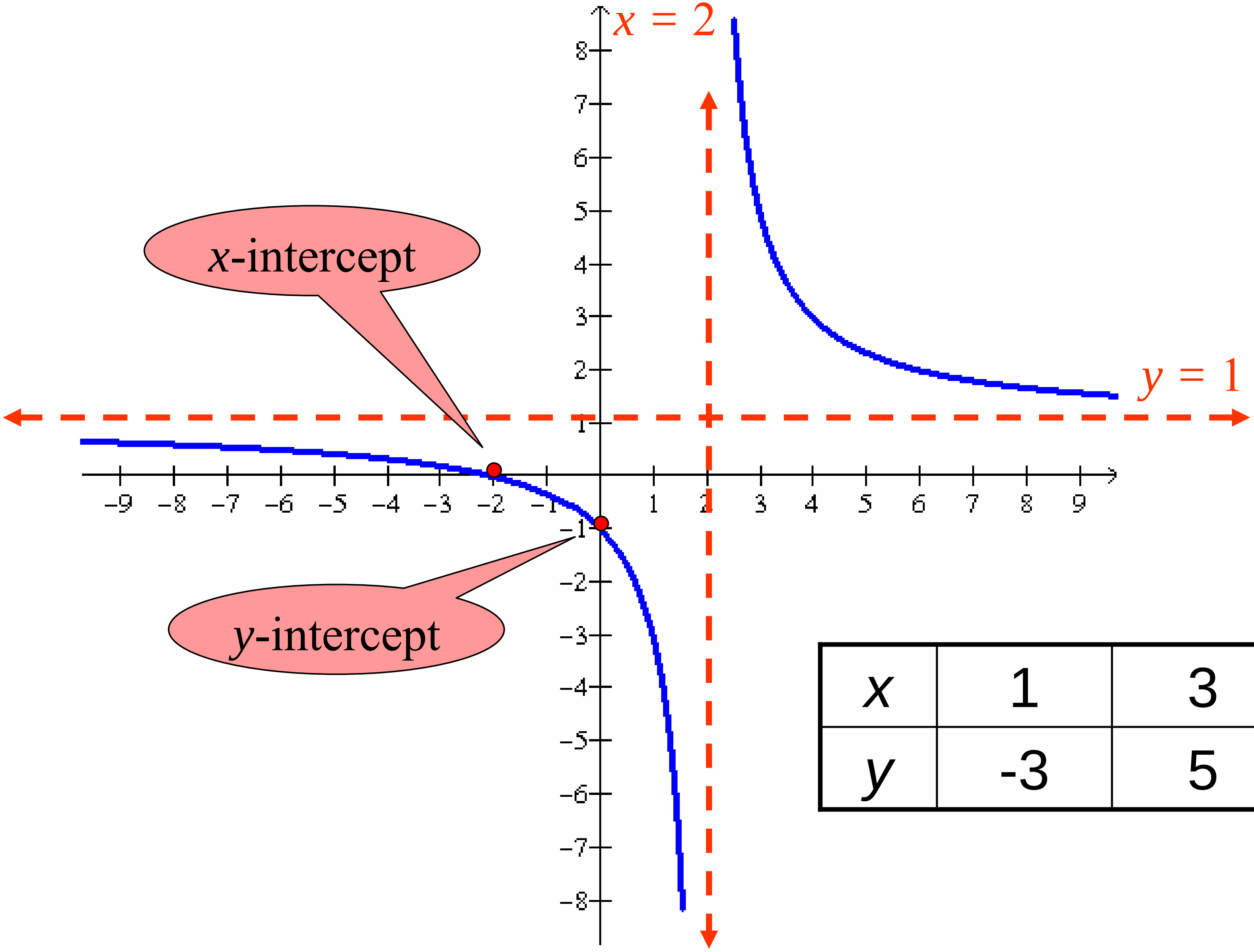
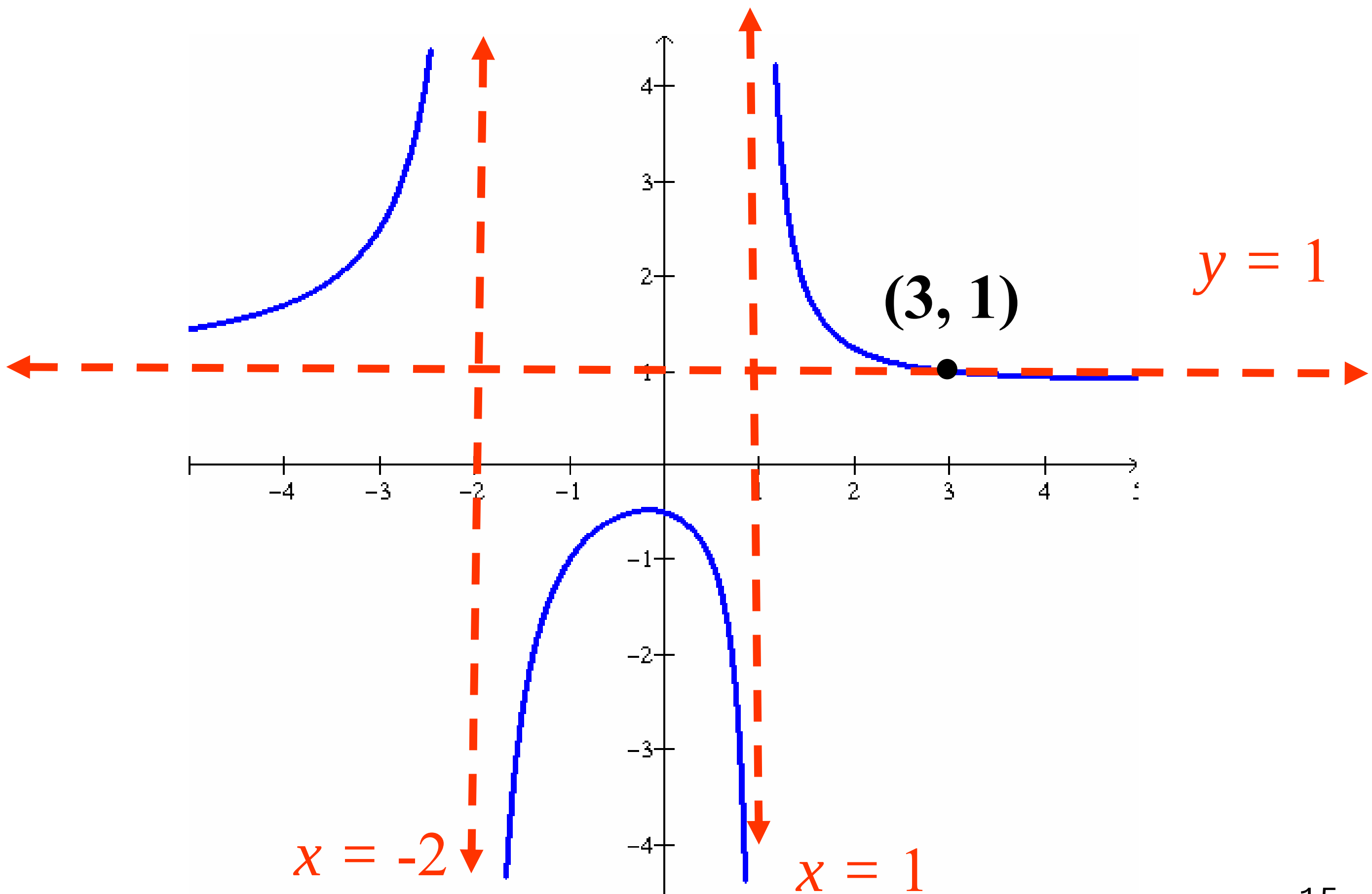


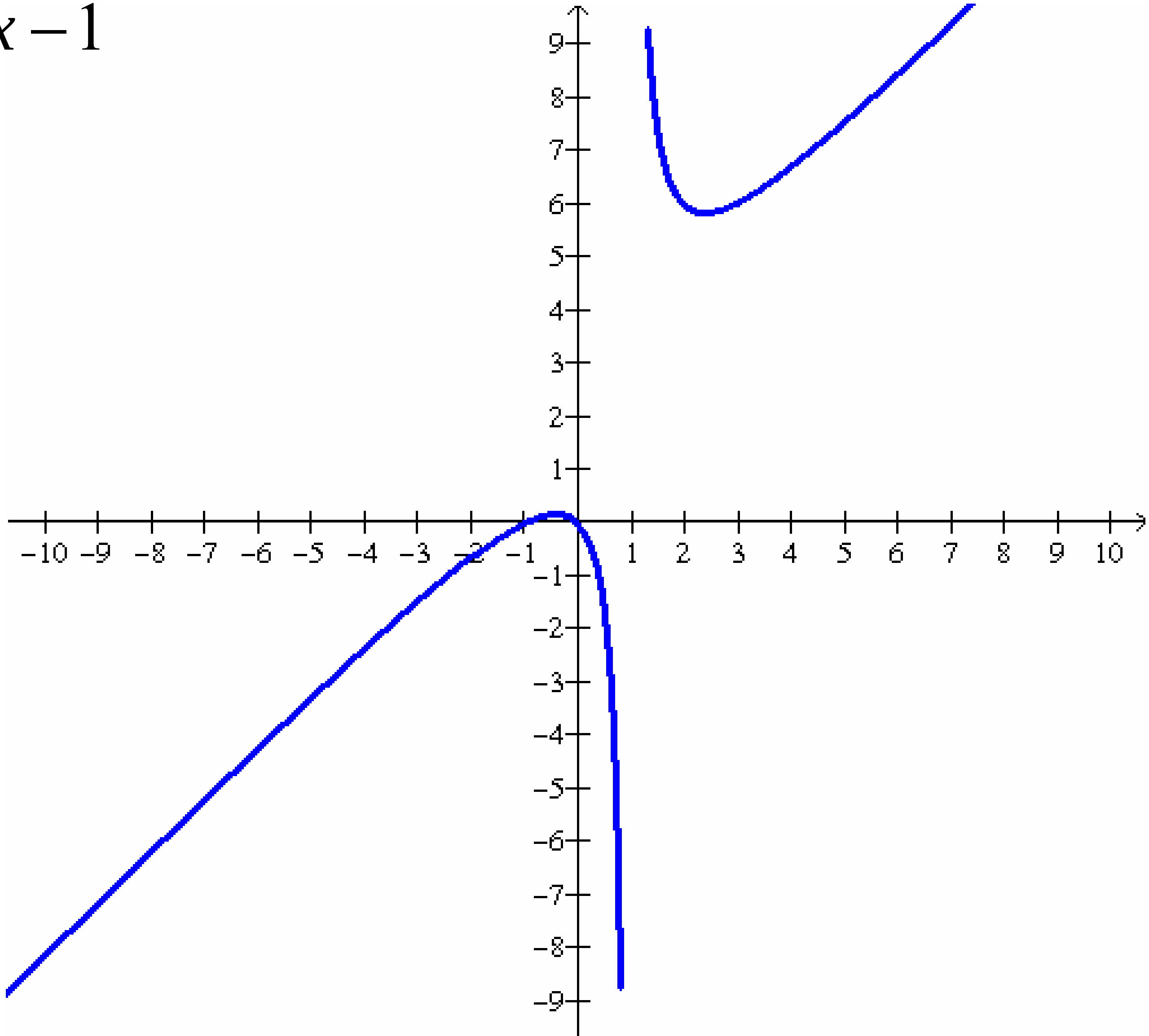
x	1	3
y	-3	5

<u>Ex8:</u> Sketch the graph of the following functions.

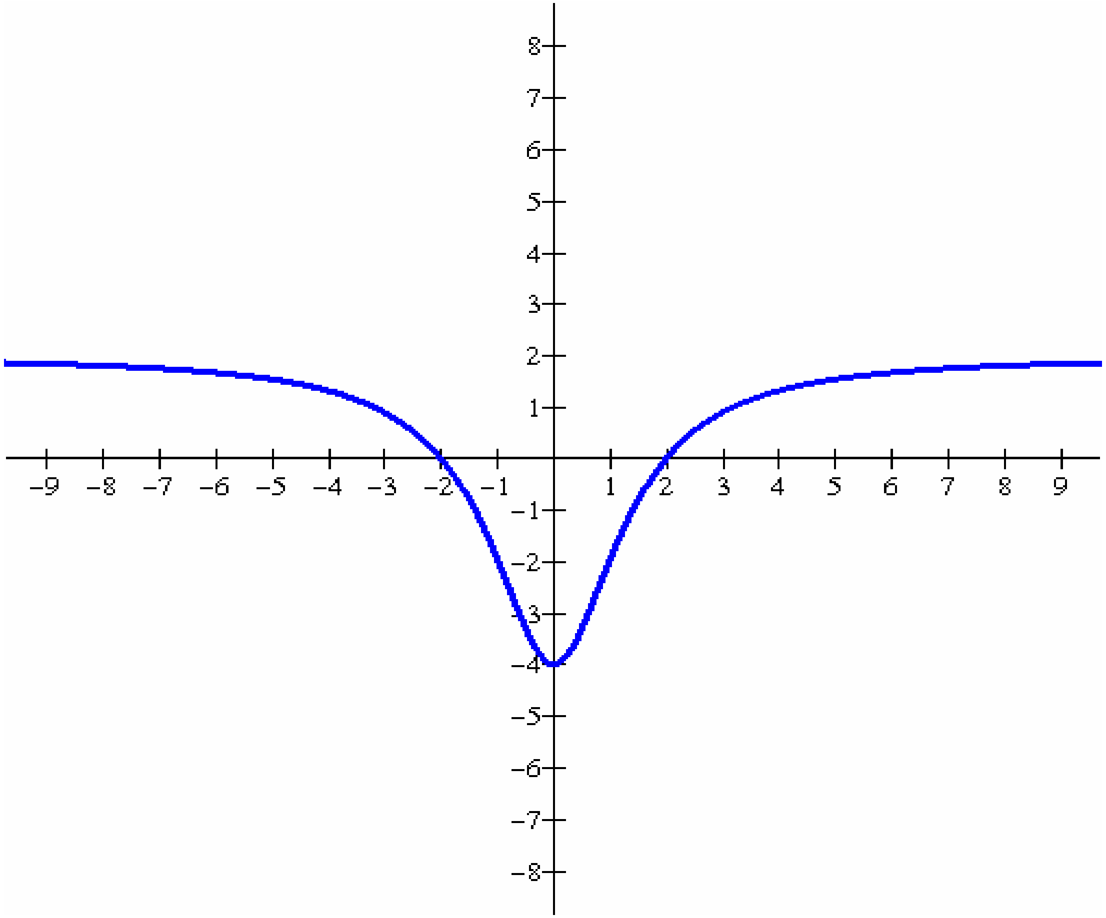
$$a) \quad f(x) = \frac{x^2 + 1}{x^2 + x - 2}$$



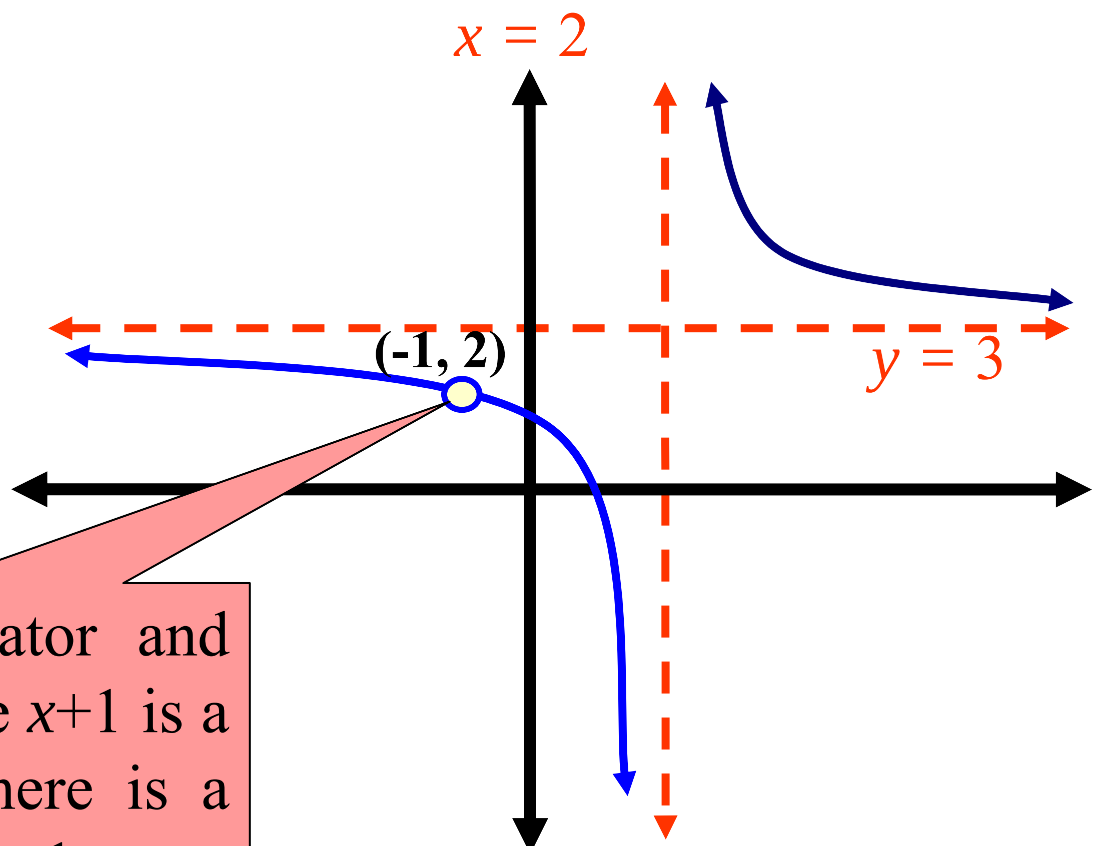
$$b)\ g(x)=\frac{x^2+x}{x-1}$$



$$c)\ \ h(x)=\frac{2x^2-8}{x^2+2}$$



$$d)\ k(x) = \frac{3x^2 - 3}{x^2 - x - 2}$$



Factor the numerator and denominator. Since $x+1$ is a common factor, there is a hole in the graph at -1 .

Ex9:

Determine the point at which the graph

$$F(x) = \frac{3x^3 + 2x^2 - 8x - 12}{x^2 + 4}$$

intersects its slant asymptote.