2.2 INTRODUCTION TO FUNCTIONS

Definition of a Relation

A ***relation*** is any set of ordered pairs.

- The set of all first coordinate of the ordered pairs is called the ***domain*** **(المجال)** of the relation,
- The set of all second coordinate of the ordered pairs is called the ***range*** **(المدى)** of the relation.

***Ex1*:** Consider the function {(0,0) , (-1,1), (2,3) }

The domain is {0,-1,2}

The range is {0,1,3}

Definition of a Function (دالة)

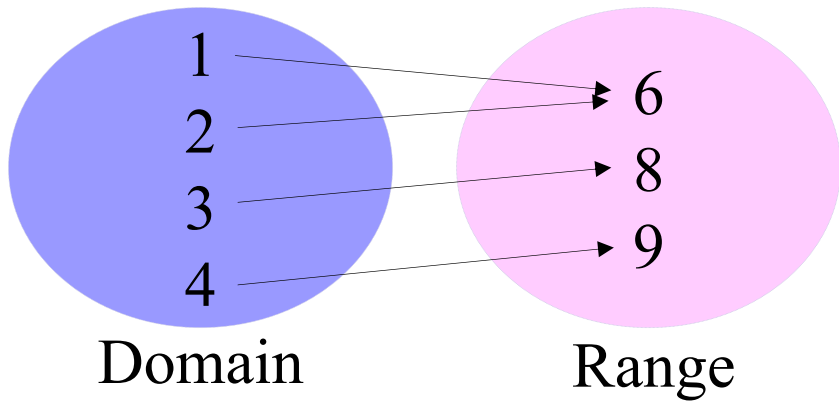
A ***function*** is a set of ordered pairs in which no two ordered pairs have the same first coordinate and different second coordinates.

- Such that each element in the domain corresponds to *exactly one* element in the range.

***Ex2*:** Determine whether each relation defines *y* as a function of *x* :

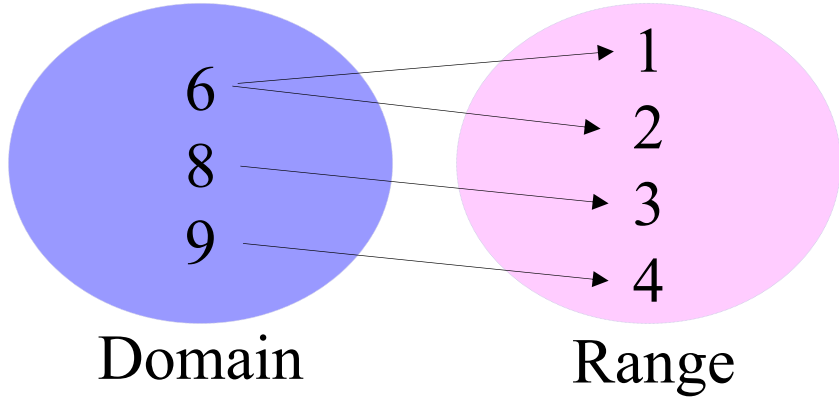
a) {(1, 6), (2, 6), (3, 8), (4, 9)}

Every element in the domain corresponds to exactly one element in the range



b) {(6, 1), (6, 2), (8, 3), (9, 4)}

6 in the domain corresponds to both 1 and 2 in the range.



Functions are usually given in terms of equations rather than as sets of ordered pairs where:

- A variable *y* is said to be a function of the variable *x* if for each value of *x*, there is one and only one value of *y*.
- The variable *x* is called the ***independent variable*** because it can be assigned any value from the domain.
- The variable *y* is called the ***dependent variable*** because its value depends on *x*.

How to identify if an equation represents a function or not?

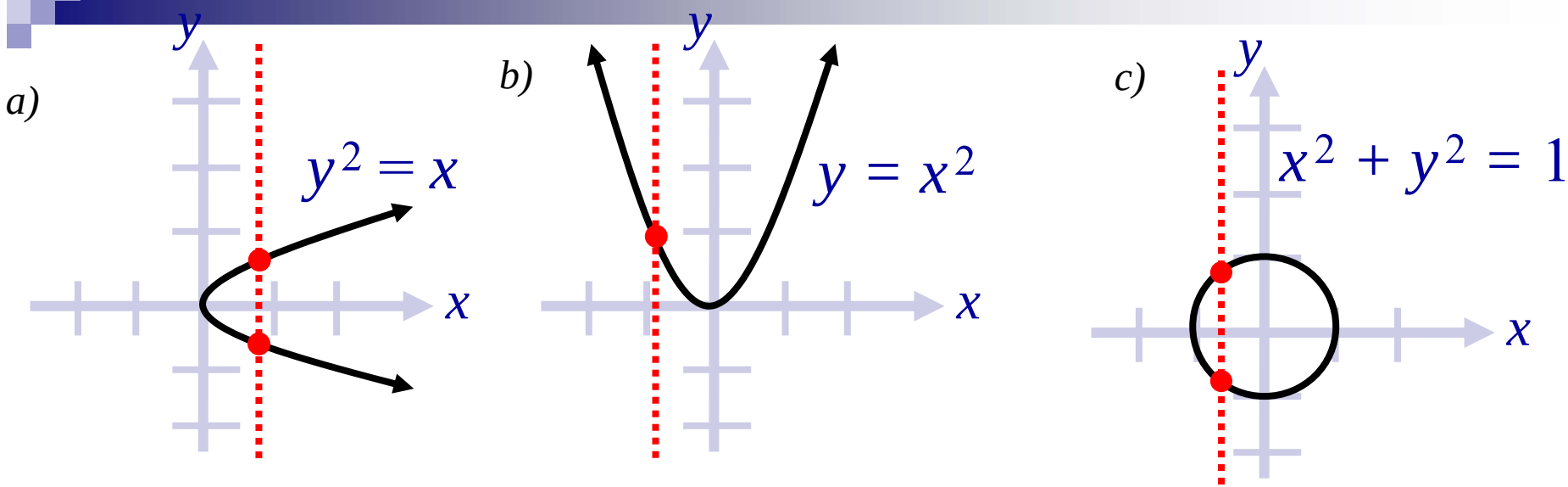
<u>Case1:</u>Algebraically جبرياً

Solve for y

- one value of y ⇒ function
- two values of y ⇒ not function

<u>Case 2:</u>Graphically من الرسم

The vertical (العمودي) line test : A graph is the graph of a function if and only if no vertical line intersects the graph at more than one point.



Not function

function

Not function

Function Notation

Functions represented by equations are often named using a letter such as *f* or *g*.

The symbol ***f* (*x*),** read as "the value of *f* at *x*" or simply as "*f* of *x*", is the element in the range of *f* that corresponds with the domain element *x*. That is, $\boldsymbol{y = f(x)}$

***Ex3*:** Determine whether each equation defines y as a function of x :

Solution

$$a)\ x^2 + y = 4$$

$$x^2 + y - x^2 = 4 - x^2$$

$$y = 4 - x^2$$

For each value of x there is one and only one value of y.
Therefore, the equation defines y as a function of x.

$$b)\ x^2 + y^2 = 4$$

$$y^2 = 4 - x^2$$

$$y = \pm\sqrt{4 - x^2}$$

The $\pm$ in the resulting equation shows that for certain values of x (all values between -2 and 2), there are two values of y.
Therefore, the equation does not define y as a function of x.

To ***evaluate*** a function $f(x)$ at $x = a$, substitute the specified value a for x into the given function.

Ex.4 If $f(x) = x^2 + 3x + 5$, evaluate:

a) $f(2)$

b) $f(x + 3)$

c) $f(-x)$

Solution

a) $f(2) = 2^2 + 3 \cdot 2 + 5$

$f(2) = 15$

b) $f(x + 3)$

$$f(x + 3) = (x + 3)^2 + 3 \cdot (x + 3) + 5$$

$$= x^2 + 6x + 9 + 3x + 9 + 5$$

$$= x^2 + 9x + 23$$

c) $f(-x)$

$$f(-x) = (-x)^2 + 3(-x) + 5$$

$$= x^2 - 3x + 5$$

Ex: Identify which one of the following relations define y as a function of x

$a)\ x^2 + y^2 = 4$ $\qquad$ $c)\ y^3 = x^2$

$b)\ 3x + y = -2$ $\qquad$ $d)\ |y| = x$

Ex: Let $f(x) = 2x^2 - 2$. Find

$a.\ f(-1)$

$b.\ f(0)$

$c.\ f(2a)$

Piecewise-defined functions

are functions represented by more than one expression.

Ex.5: Given the function:

$$f(x) = \begin{cases} 2x, & x < -2 \\ x^2, & -2 \le x < 1 \\ 4 - x, & x \ge 1 \end{cases}$$

Find: *a*) $f(-4)$, $f(-1)$, $f(7)$.

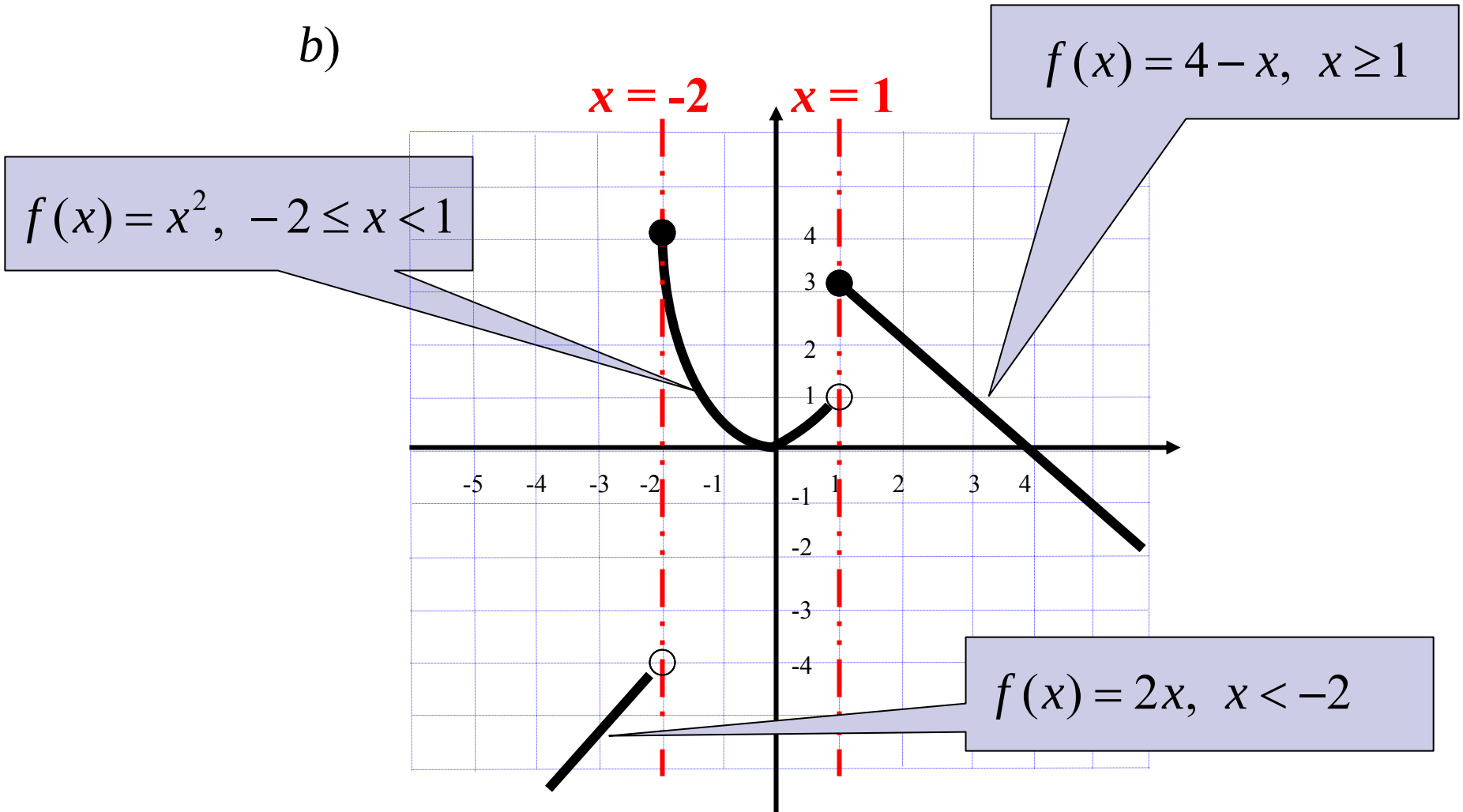
b) Graph the function.

Solution Find: *a*) $f(-4) = 2(-4) = -8$

$f(-1) = (-1)^2 = 1$

$f(7) = 4 - 7 = -3$

b)



Ex:

$$\text{Let } f(x) = \begin{cases} 2 & \text{if} \quad x < -2 \\ |x| & \text{if} \quad -2 \le x < 1 \\ x-1 & \text{if} \quad 1 \le x < 5 \end{cases}$$

1) sketch the graph of $f(x)$

2) find $f(-2),\ f(0),\ f(3),\ f(6)$

The domain of a function *f*

Is the set of all real numbers for which the function makes sense.

Finding a Function's Domain

- Exclude from a function's domain real numbers that cause ***division by zero*** and real numbers that ***result in an even root of a negative number.***

Ex.6 Find the domain of each function:

a) $f(x) = x^2 - 7x$ contains neither division nor an even root.

The domain of f is the set of all real numbers.

b) $g(x) = \dfrac{6x}{x^2 - 9}$ we exclude values that cause $x^2 - 9$ to equal 0.

The domain of g is $\{x \mid x \neq -3, x \neq 3\}$.

c) $h(x) = \sqrt{3x + 12}$ contains an even root

$$3x + 12 \geq 0$$

$$3x \geq -12$$

$$x \geq -4$$

The domain of h is $\{x \mid x \geq -4\}$ or the interval $[-4, \infty)$.

Ex: Find the domain of the following functions

1) $f(x) = 3x + 5$

2) $f(x) = \dfrac{1}{x+2}$

3) $f(x) = \sqrt{x+2}$

4) $g(x) = \sqrt{4 - x^2}$

5) $h(x) = \sqrt{x - x^2}$

6) $f(x) = \sqrt{|x| - 1}$

7) $w(x) = \sqrt[3]{x+1}$

8) $f(x) = \sqrt{\dfrac{x-1}{(x-2)(3-x)}}$

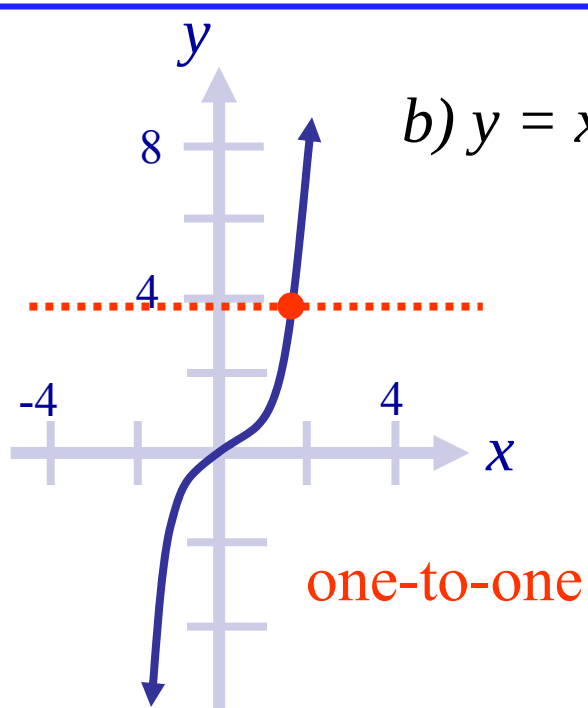
The **graph of a function** is the graph of its ordered pairs.

- The **zeros of a function**, f, are the x-values for which $f(x) = 0$.
- A **function can have more that one x-intercept but at most one y-intercept**, $f(0) = b$.

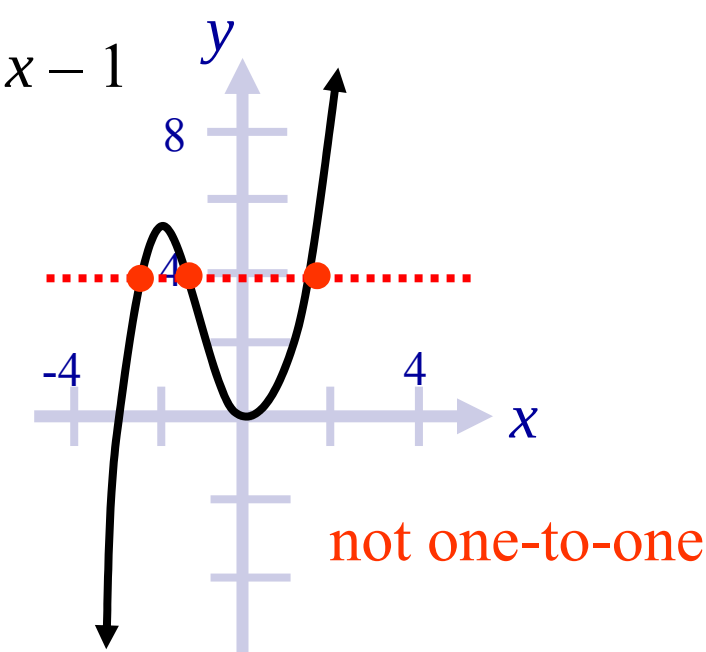
Horizontal (أفقي) Line Test

- A function y = f (x) is one-to-one if and only if no horizontal line intersects the graph of y = f (x) in more than one point.

a) $y = x^3$



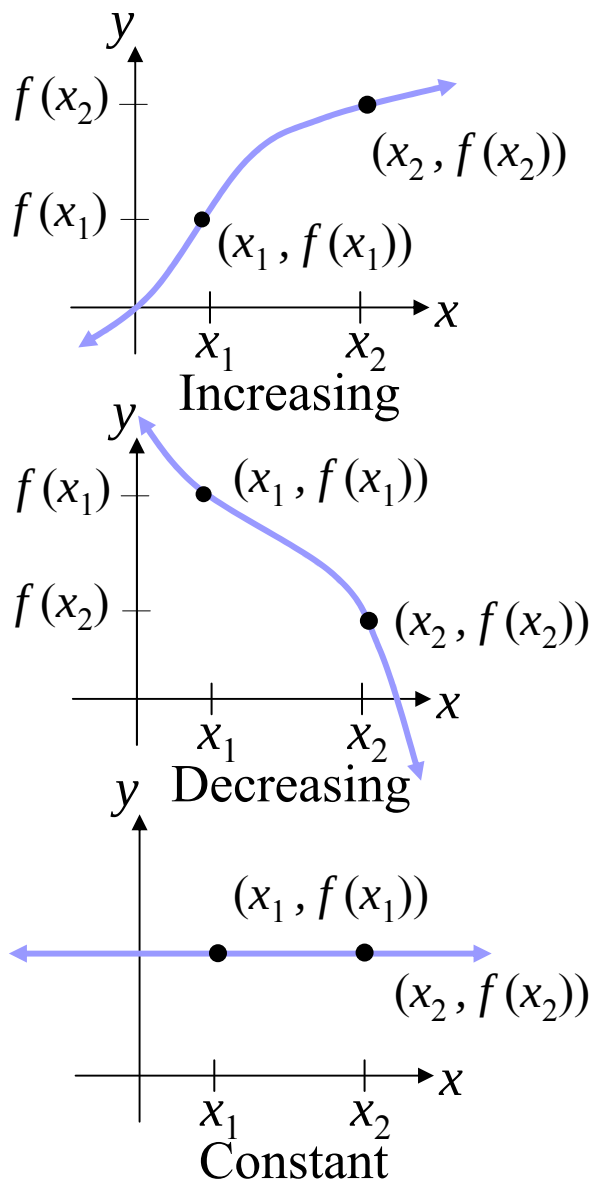
b) $y = x^3 + 3x^2 - x - 1$



Increasing, Decreasing, and Constant Functions

تزايد تناقص ثابت

1. A function is ***increasing*** on an interval
   - if for any x_1 and x_2 in the interval,
   - where $x_1 < x_2$,
   - then $f(x_1) < f(x_2)$.
2. A function is ***decreasing*** on an interval
   - if for any x_1 and x_2 in the interval,
   - where $x_1 < x_2$,
   - then $f(x_1) > f(x_2)$.
3. A function is ***constant*** on an interval
   - if for any x_1 and x_2 in the interval,
   - where $x_1 < x_2$,
   - then $f(x_1) = f(x_2)$.

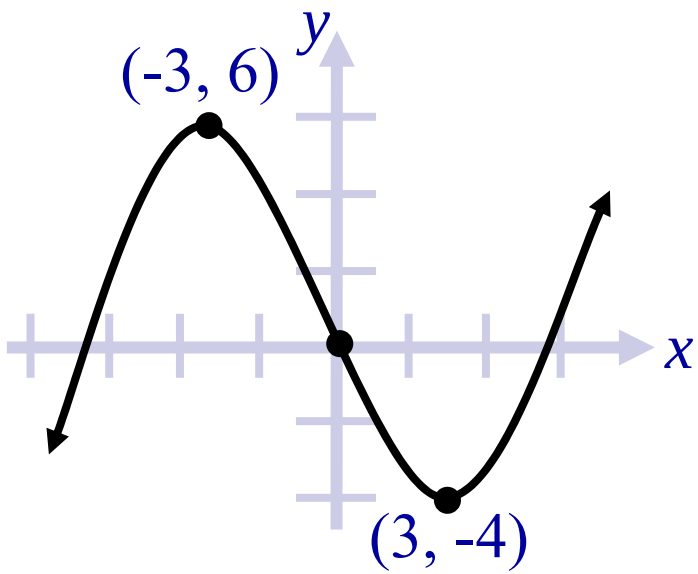


Ex 7: Consider the graph of the function *f(x)*

Rules:

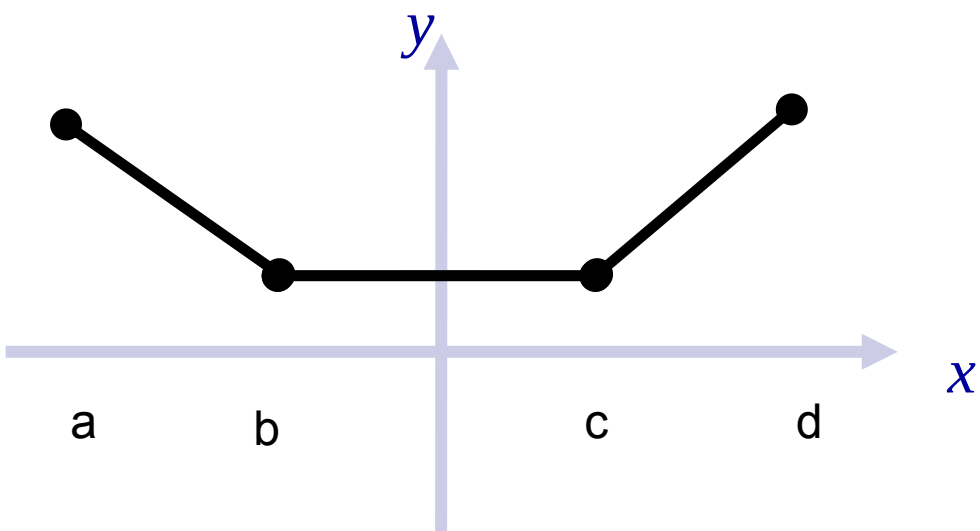
1) Start from left to right

2) Take the intervals from the *x*-axis



- **Increasing** on $(-\infty, -3] \cup [3, +\infty)$.
- **Decreasing** on $[-3, 3]$

Ex 8:



Rules:

1) Start from left to right

2) Take the intervals from the *x*-axis

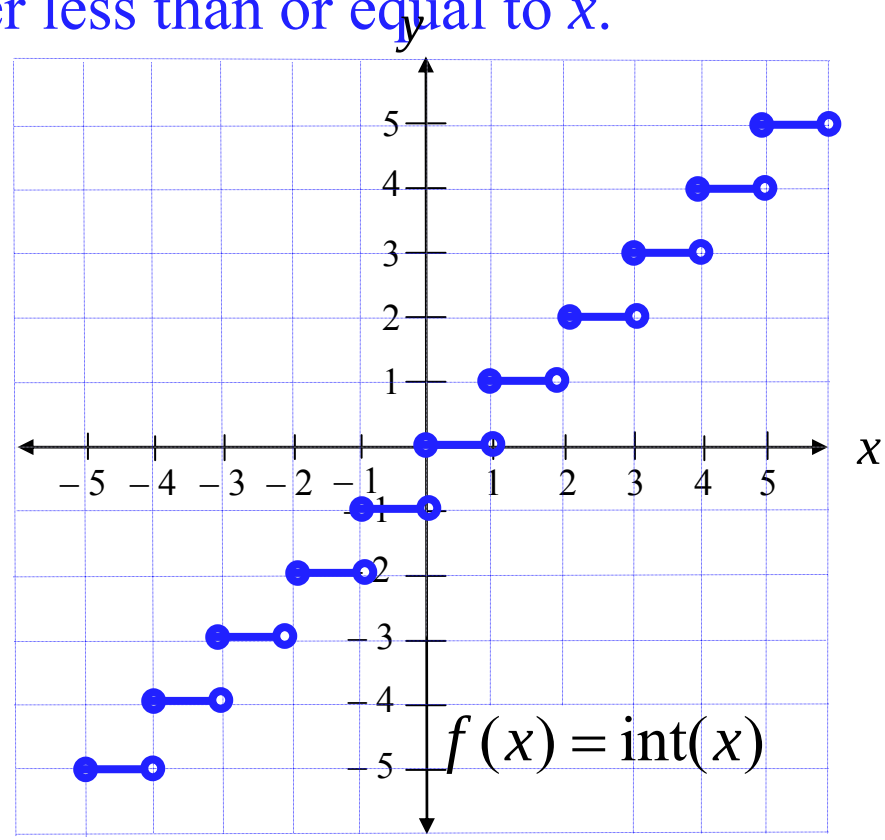
- **Increases** on [c, d]
- **Decreases** on [a, *b*]
- **Constant** on [b, *c*]

The Greatest Integer Function

$f(x) = \text{int}(x)$ or $[[x]]$ greatest integer less than or equal to x.

If $2 \le x < 3$, then $\text{int}(x) = 2$
If $1 \le x < 2$, then $\text{int}(x) = 1$
If $0 \le x < 1$, then $\text{int}(x) = 0$
If $-1 \le x < 0$, then $\text{int}(x) = -1$
If $-2 \le x < -1$, then $\text{int}(x) = -2$
If $-3 \le x < -2$, then $\text{int}(x) = -3$

In general, where n is an integer,
If $n \le x < n+1$, then $\text{int}(x) = n$



- If $[f(x)] = n \Leftrightarrow n \le f(x) < n+1, n:\text{integer}$
- If $f(x) = [x]$, then

$D_f = (-\infty, \infty)$

$R_f =$ the set of all integers

Ex.9 int(4.5) = 4
int(–3.5) = – 4
int(3) = 3
int(0.7) = 0

Ex. Evaluate:

1) $[1.5]$
2) $[1.7]$
3) $[1.3]$
4) $[-1.5]$
5) $[-1.7]$
6) $[-1.3]$
7) $[2]$
8) $[-2]$
9) $[0]$
10) $\dfrac{[\pi]}{[-\pi]}$
11) *f(-5)+f(5)*

If $f(x) = \begin{cases} \left[\dfrac{1}{3}x\right] & if \quad x < 0 \\ |4 - 3x| & if \quad x \geq 0 \end{cases}$

Ex. Solve the following equations

1) $[2x-1]=2$

2) $[2x-1]=\frac{1}{2}$

Ex. Find the domain of the function $f(x)=\frac{1}{[x-1]}$

Ex. Find the x-intercept and y-intercept of $f(x)=\left[\frac{2x+1}{3}\right]$

Ex.

$$\text{Let } f(x)=\begin{cases} 3 & if \quad x<-2 \\ x^2 & if \quad -2\le x<1 \\ -x+1 & if \quad 1\le x<5 \end{cases}$$

1) sketch the graph of $f(x)$

2) find $f(-2), f(0), f(3), f(6)$

3) determine the $x-$intercept and the $y-$intercept

4) find the interval(s) of increasing, decreasing and constant

5) find the domain and the range of $f(x)$

6) find all values of x such that $f(x)=4$