Math 001 Inverse Function

Notes :

1. f - inverse $= f^{-1} \neq \dfrac{1}{f}$ = reciprocal of f
2. If $f(x)$ is not a $1-1$ function, then $f^{-1}(x)$ doesn't exist.
3. If $f(x)$ is a $1-1$ function, then $f^{-1}(x)$ exists and it is $1-1$.
4. $f(a) = b \iff f^{-1}(b) = a$
5. $\mathrm{D}_f = \mathrm{R}_{f\text{-}1}$ and $\mathrm{R}_f = \mathrm{D}_{f\text{-}1}$
6. $(f \circ f^{-1})(x) = x, \ \forall x \in \mathrm{D}_{f\text{-}1}$ and $(f^{-1} \circ f)(x) = x, \ \forall x \in \mathrm{D}_f$
7. The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetric with respect to the line $y = x$.
8. If f and g are $1-1$ functions, then $(f \circ g)^{-1}(x) = g^{-1}(x) \circ f^{-1}(x)$

Hints :

1. To check that the functions $f(x)$ and $g(x)$ are inverses of each other :
   (a) $f(x)$ and $g(x)$ must be $1-1$ functions.
   (b) $(f \circ g)(x)$ must be x (and also $(g \circ f)(x) = x$)
2. To find the inverse of a function $f(x)$:
   (a) $f(x)$ must be $1-1$.
   (b) Interchange x with y in the equation of $f(x)$.
   (c) Solve for y. This y is $f^{-1}(x)$.

Solved Problems :

1. If $f(x)$ is $1-1$ and $f(-2) = 3$, then find the value of $5f^{-1}(3) + 2(f \circ f^{-1})(-1) - (f^{-1} \circ f)(4)$.
   Ans. : $= 5(-2) + 2(-1) - (4) = -10 - 2 - 4 = -16$
2. If $f = \{(-1,3), (2,4), (3,5)\}$, find f^{-1}
   Ans. : $f^{-1} = \{(3,-1), (4,2), (5,3)\}$
3. Which of the following functions is not $1-1$?
   (a) $f(x) = x^2 + 4, \ x \geq 0$ (b) $f(x) = |7x + 14|, \ x \geq -2$
   (c) $f(x) = 3$ (d) $f(x) = (x+1)^2 - 1, \ -2 \leq x < 2$
   Ans. : Use the graph to see that the functions in parts (a), (b) and (d) are $1-1$, while the function $f(x) = 3$ is not $1-1$ (its graph is a horizental line).
   Notice that $f^{-1}(x)$ exists in parts (a), (b) and (d), while $f^{-1}(x)$ does not exist in part (c).
4. Find $f^{-1}(x), \mathrm{D}_{f\text{-}1}$ and $\mathrm{R}_{f\text{-}1}$ of the following functions?
   (a) $f(x) = -\sqrt{x-2}, \ x \geq 2$ (b) $f(x) = |4 - x|, \ x \leq 4$ (c) $f(x) = x^2 - 4x, \ x \leq 2$
   Ans. : Notice that each one of the above functions is $1-1$ (check the graphs)
   (a) $y = f(x) = -\sqrt{x-2}$, notice that $y \leq 0$. To find $f^{-1}(x)$:
   interchange x with y to get : $x = -\sqrt{y-2}$, where $x \leq 0$. Next solve for y :
   $x^2 = y - 2 \implies y = f^{-1}(x) = x^2 + 2, \ x \leq 0$.
   $\mathrm{D}_{f\text{-}1} = \mathrm{R}_f = (-\infty, 0)$ and $\mathrm{R}_{f\text{-}1} = \mathrm{D}_f = [2, \infty)$

(b) $y = |4-x|, x \le 4$. Notice that $y \ge 0$ and $y = |4-x| = 4-x$, because $x \le 4$.
To find $f^{-1}(x)$: interchange x with y to get : $x = 4-y$, where $x \ge 0$. Next solve for y :
$y = f^{-1}(x) = 4-x,\ x \ge 0$. $\mathrm{D}_{f\text{-}1} = \mathrm{R}_f = [0,\infty)$ and $\mathrm{R}_{f\text{-}1} = \mathrm{D}_f = (-\infty, 4]$

(c) $y = x^2 - 4x,\ x \le 2$. Now $y = (x^2-4x+4)-4 = (x-2)^2-4,\ x \le 2$, notice that $y \ge -4$.
Interchange x with y to get : $x = (y-2)^2 - 4$, where $y \le 2$. Next solve for y :
$x+4 = (y-2)^2 \implies y-2 = \pm\sqrt{x+4} \implies y = f^{-1}(x) = 2-\sqrt{x+4}$, because $y \le 2$.
$\mathrm{D}_{f\text{-}1} = \mathrm{R}_f = [-4,\infty)$ and $\mathrm{R}_{f\text{-}1} = \mathrm{D}_f = (-\infty, 2]$

5. Find $f^{-1}(x)$ if $f(x) = \dfrac{3x-1}{2x}$.
Ans. : For each value of y, there's only one value for x, so $f(x)$ is a $1-1$ function.
To find $f^{-1}(x)$: interchange x with y to get : $x = \dfrac{3y-1}{2y} \implies 2xy = 3y-1$
$\implies 1 = 3y - 2xy = (3-2x)y \implies y = f^{-1}(x) = \dfrac{1}{3-2x}$

6. If $f(x) = x^2+1,\ x \le 0$, then find the value of : (a) $(f \circ f^{-1} \circ f)(-4)$ (b) $f^{-1}(2)$
Ans. : With this particular domain, $f(x)$ is $1-1$. There's no need to find $f^{-1}(x)$.
(a) $(f \circ f^{-1} \circ f)(-4) = f(-4) = 16+1 = 17$
(b) To find $f^{-1}(2)$, let $y = 2$ in the equation of $f(x)$, so $2 = x^2+1 \implies x^2 = 1 \implies x = \pm 1$, choose $x = -1$ (because $x \le 0$) $\implies f^{-1}(2) = -1$

7. If $f(x) = 5x-4$, $g(x) = \frac{x}{5}+2$, find $(f \circ g)^{-1}(x)$.
Ans. : First $(f \circ g)(x) = f(g(x)) = f(\frac{x}{5}+2) = 5(\frac{x}{5}+2) - 4 = x+10-4 = x+6$, which's $1-1$.
To find the inverse : interchange x with y to get : $x = y+6 \implies y = (f \circ g)^{-1}(x) = x-6$.
Another way to find $(f \circ g)^{-1}(x)$ is by using : $(f \circ g)^{-1}(x) = g^{-1}(x) \circ f^{-1}(x)$ (try !)

8. If the functions $f(x) = ax+1$ and $g(x) = 3x+b$ are inverses of each other, then find $a+b$.
Ans. : $(f \circ g)(x) = x \implies f(3x+b) = a(3x+b)+1 = x \implies 3ax + (ab+1) = x \equiv (1)x+0 \implies 3a = 1$ and $ab+1 = 0 \implies a = \frac{1}{3}$, $b = -3 \implies a+b = -\frac{8}{3}$.

By : A. ALSHALLALI