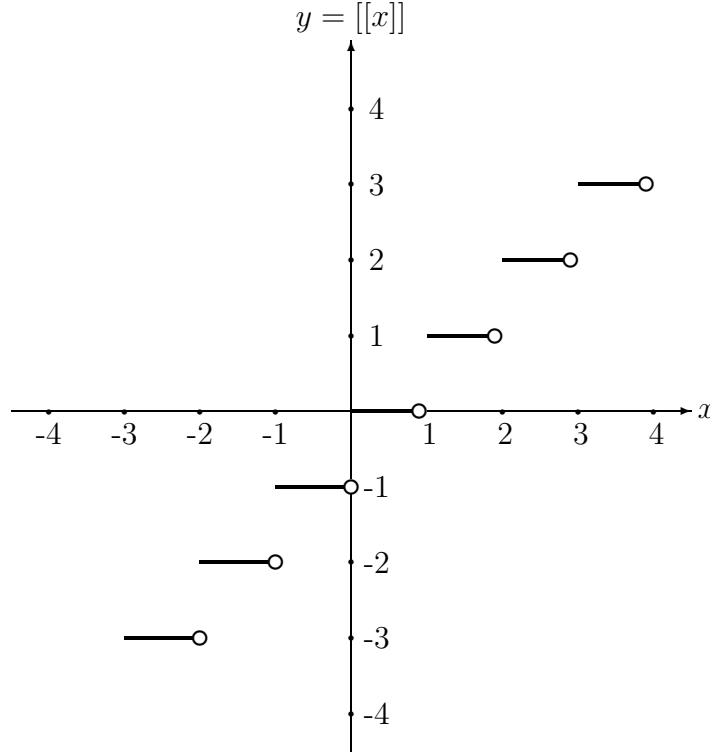


# Greatest Integer Function

**Definition:** The greatest integer function  $y = \llbracket x \rrbracket$  is the greatest integer less than or equal to  $x$ .

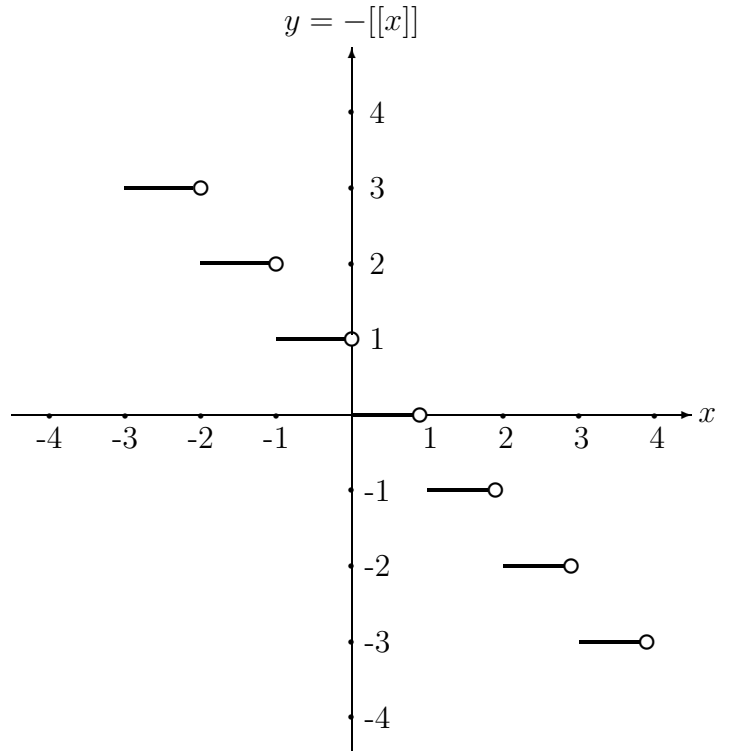
**Example:**  $\llbracket 0 \rrbracket = \llbracket 0.01 \rrbracket = \llbracket 0.5 \rrbracket = \llbracket .9 \rrbracket = \llbracket 0.99 \rrbracket = 0$ ,  $\llbracket 1 \rrbracket = \llbracket 1.4 \rrbracket = \llbracket 1.98 \rrbracket = 1$ ,  $\llbracket -1 \rrbracket = \llbracket -1.95 \rrbracket = \llbracket -0.01 \rrbracket = -1$ ,  $\llbracket 7.8 \rrbracket = 7$ ,  $\llbracket -39.6 \rrbracket = -40$ ,  $\llbracket \pi \rrbracket = 3$ ,  $\llbracket -\sqrt{2} \rrbracket = -2$ ,  $\llbracket -\sqrt{8} \rrbracket = -3$

$y = \llbracket x \rrbracket$	
$x$	$y = \llbracket x \rrbracket$
$\vdots$	$\vdots$
$-3 \leq x < -2$	-3
$-2 \leq x < -1$	-2
$-1 \leq x < 0$	-1
$0 \leq x < 1$	0
$1 \leq x < 2$	1
$2 \leq x < 3$	2
$3 \leq x < 4$	3
$\vdots$	$\vdots$



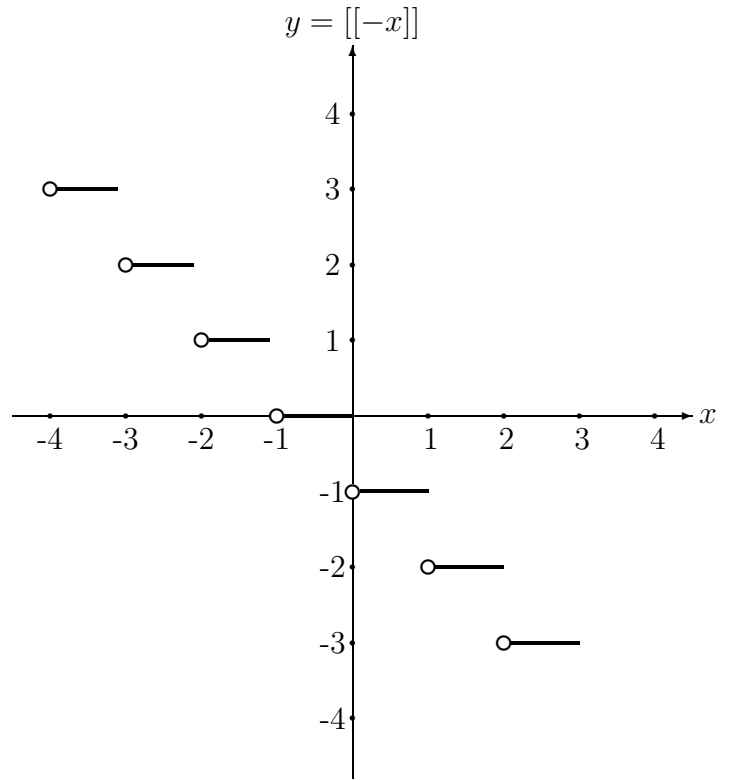
$y = -\llbracket x \rrbracket$		
$x$	$\llbracket x \rrbracket$	$y = -\llbracket x \rrbracket$
$\vdots$	$\vdots$	$\vdots$
$-3 \leq x < -2$	-3	3
$-2 \leq x < -1$	-2	2
$-1 \leq x < 0$	-1	1
$0 \leq x < 1$	0	0
$1 \leq x < 2$	1	-1
$2 \leq x < 3$	2	-2
$3 \leq x < 4$	3	-3
$\vdots$	$\vdots$	$\vdots$

Note: This graph can be obtained by reflecting the graph of  $y = \llbracket x \rrbracket$  about the x-axis.

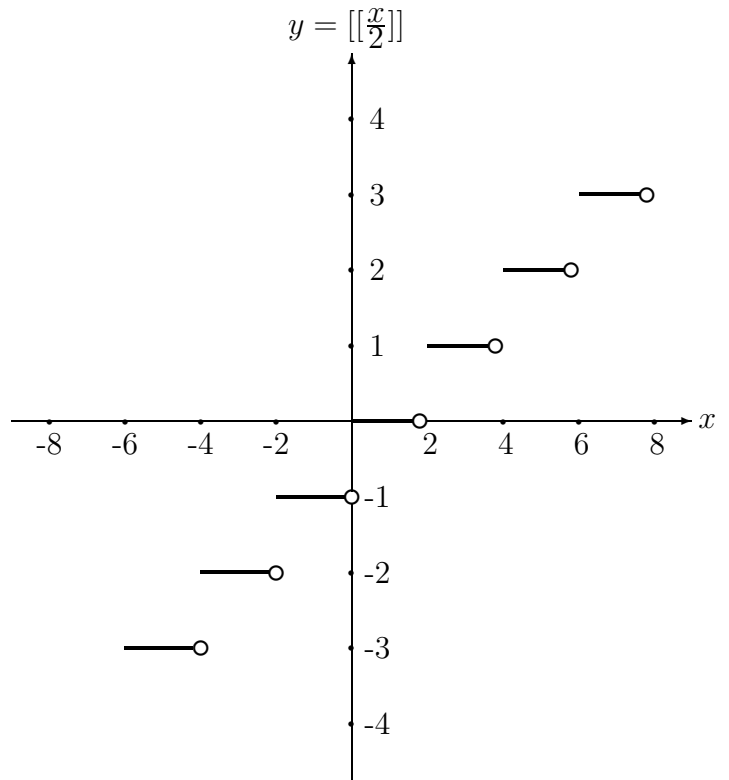


$y = \lceil \lceil -x \rceil \rceil$		
$-x$	$x$	$y = \lceil \lceil -x \rceil \rceil$
$\vdots$	$\vdots$	$\vdots$
$-3 \leq -x < -2$	$3 \geq x > 2$	$-3$
$-2 \leq -x < -1$	$2 \geq x > 1$	$-2$
$-1 \leq -x < 0$	$1 \geq x > 0$	$-1$
$0 \leq -x < 1$	$0 \geq x > -1$	$0$
$1 \leq -x < 2$	$-1 \geq x > -2$	$1$
$2 \leq -x < 3$	$-2 \geq x > -3$	$2$
$3 \leq -x < 4$	$-3 \geq x > -4$	$3$
$\vdots$	$\vdots$	$\vdots$

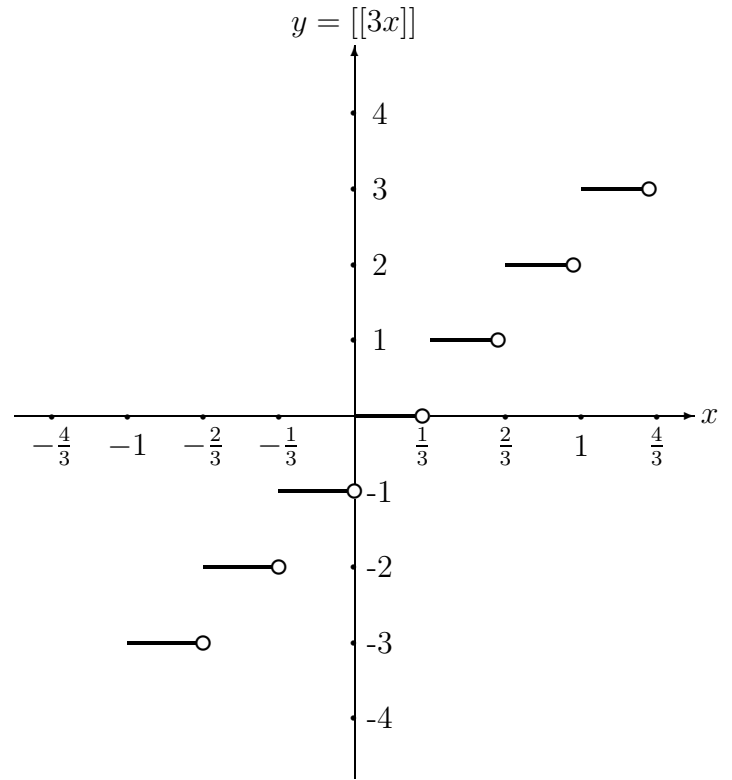
Note: This graph can be obtained by reflecting the graph of  $y = \lceil \lceil x \rceil \rceil$  about the y-axis.



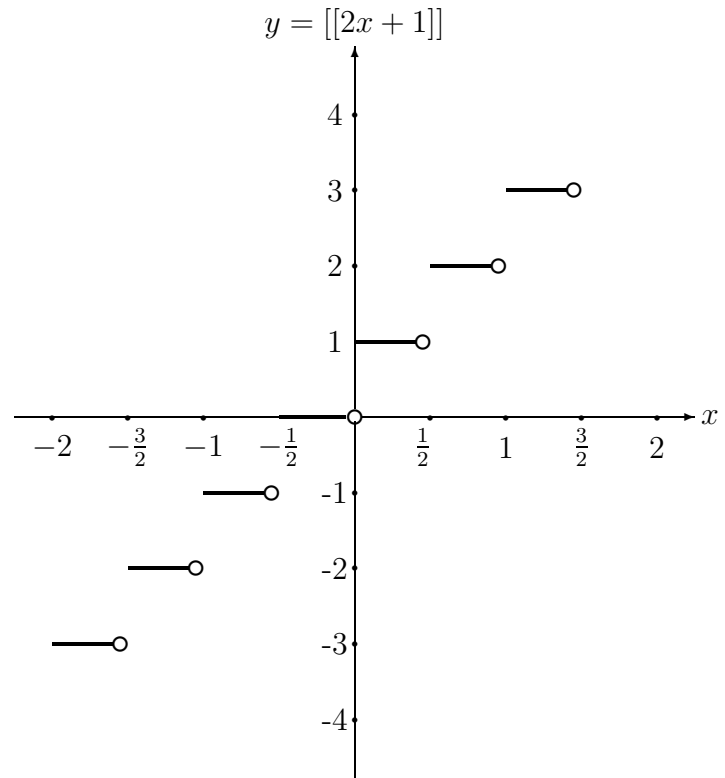
$y = \lceil \lceil \frac{x}{2} \rceil \rceil$		
$\frac{x}{2}$	$x$	$y = \lceil \lceil \frac{x}{2} \rceil \rceil$
$\vdots$	$\vdots$	$\vdots$
$-3 \leq \frac{x}{2} < -2$	$-6 \leq x < -4$	$-3$
$-2 \leq \frac{x}{2} < -1$	$-4 \leq x < -2$	$-2$
$-1 \leq \frac{x}{2} < 0$	$-2 \leq x < 0$	$-1$
$0 \leq \frac{x}{2} < 1$	$0 \leq x < 2$	$0$
$1 \leq \frac{x}{2} < 2$	$2 \leq x < 4$	$1$
$2 \leq \frac{x}{2} < 3$	$4 \leq x < 6$	$2$
$3 \leq \frac{x}{2} < 4$	$6 \leq x < 8$	$3$
$\vdots$	$\vdots$	$\vdots$



$y = \lceil\lceil 3x \rceil\rceil$		
$3x$	$x$	$y = \lceil\lceil 3x \rceil\rceil$
$\vdots$	$\vdots$	$\vdots$
$-3 \leq 3x < -2$	$-1 \leq x < -\frac{2}{3}$	-3
$-2 \leq 3x < -1$	$-\frac{2}{3} \leq x < -\frac{1}{3}$	-2
$-1 \leq 3x < 0$	$-\frac{1}{3} \leq x < 0$	-1
$0 \leq 3x < 1$	$0 \leq x < \frac{1}{3}$	0
$1 \leq 3x < 2$	$\frac{1}{3} \leq x < \frac{2}{3}$	1
$2 \leq 3x < 3$	$\frac{2}{3} \leq x < 1$	2
$3 \leq 3x < 4$	$1 \leq x < \frac{4}{3}$	3
$\vdots$	$\vdots$	$\vdots$

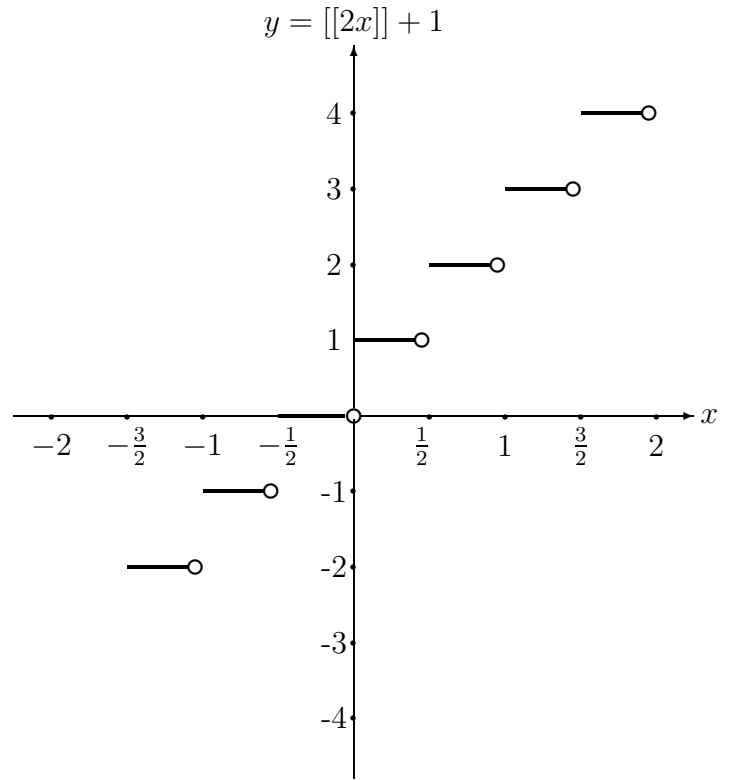


$y = \lceil\lceil 2x + 1 \rceil\rceil$		
$2x + 1$	$x$	$y = \lceil\lceil 2x + 1 \rceil\rceil$
$\vdots$	$\vdots$	$\vdots$
$-3 \leq 2x + 1 < -2$	$-2 \leq x < -\frac{3}{2}$	-3
$-2 \leq 2x + 1 < -1$	$-\frac{3}{2} \leq x < -1$	-2
$-1 \leq 2x + 1 < 0$	$-1 \leq x < -\frac{1}{2}$	-1
$0 \leq 2x + 1 < 1$	$-\frac{1}{2} \leq x < 0$	0
$1 \leq 2x + 1 < 2$	$0 \leq x < \frac{1}{2}$	1
$2 \leq 2x + 1 < 3$	$\frac{1}{2} \leq x < 1$	2
$3 \leq 2x + 1 < 4$	$1 \leq x < \frac{3}{2}$	3
$\vdots$	$\vdots$	$\vdots$



$y = \lceil 2x \rceil + 1$		
$2x$	$x$	$y = \lceil 2x \rceil + 1$
$\vdots$	$\vdots$	$\vdots$
$-3 \leq 2x < -2$	$-\frac{3}{2} \leq x < -1$	$-3 + 1 = -2$
$-2 \leq 2x < -1$	$-1 \leq x < -\frac{1}{2}$	$-2 + 1 = -1$
$-1 \leq 2x < 0$	$-\frac{1}{2} \leq x < 0$	$-1 + 1 = 0$
$0 \leq 2x < 1$	$0 \leq x < \frac{1}{2}$	$0 + 1 = 1$
$1 \leq 2x < 2$	$\frac{1}{2} \leq x < 1$	$1 + 1 = 2$
$2 \leq 2x < 3$	$1 \leq x < \frac{3}{2}$	$2 + 1 = 3$
$3 \leq 2x < 4$	$\frac{3}{2} \leq x < 2$	$3 + 1 = 4$
$\vdots$	$\vdots$	$\vdots$

Note: This graph can be obtained by translating the graph of  $y = \lceil 2x \rceil$  one unit upward.



- Notes:** 1) The domain of all the above functions =  $\mathfrak{R}$  while the range = the set of integers.  
 2)  $\lceil x \rceil = x$  if and only if  $x$  is an integer.  
 3) If  $n$  is an integer, then  $y = \lceil mx + n \rceil$  and  $y = \lceil mx \rceil + n$  have the same graph.  
 4) If  $E$  is an expression and  $n$  is an integer, then  $\lceil E \rceil = n \implies n \leq E < n + 1$ .

**Problems:** 1) Solve the equations: a)  $\lceil 2x + 1 \rceil + 1 = \frac{1}{2}$  , b)  $-2\lceil 2 - 3x \rceil - 5 = 7$

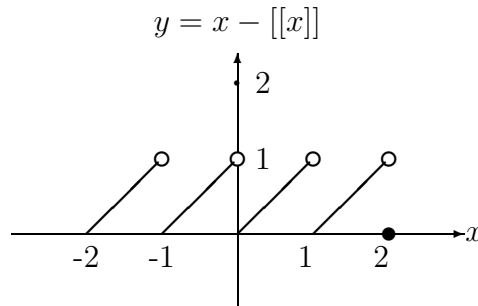
Solution: a)  $\lceil 2x + 1 \rceil + 1 = \frac{1}{2} \implies \lceil 2x + 1 \rceil = \frac{1}{2} - 1 = -\frac{1}{2}$  is impossible, because  $\lceil 2x + 1 \rceil$  must be an integer and  $-\frac{1}{2}$  is not. Therefore S.S. =  $\emptyset$

b)  $-2\lceil 2 - 3x \rceil - 5 = 7 \implies -2\lceil 2 - 3x \rceil = 12 \implies \lceil 2 - 3x \rceil = -6$   
 $\implies -6 \leq 2 - 3x < -5 \implies -8 \leq -3x < -7 \implies \frac{8}{3} \geq x > \frac{7}{3} \implies \text{S.S.} = (\frac{7}{3}, \frac{8}{3}]$

2) Graph  $y = x - \lceil x \rceil$ ,  $-2 \leq x \leq 2$ . Write the domain and range.

Solution:  $y = \begin{cases} x + 2 & \text{if } -2 \leq x < -1 \\ x + 1 & \text{if } -1 \leq x < 0 \\ x & \text{if } 0 \leq x < 1 \\ x - 1 & \text{if } 1 \leq x < 2 \\ 0 & \text{if } x = 2 \end{cases}$

Domain =  $[-2, 2]$  , Range =  $[0, 1)$



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