# Math 001 — Domain & Range

To find the domain and range of a relation or function we have to keep in mind some important notes such as :

- First The **Domain** : The domain of a function $y = f(x)$ is the set of all possible real values of $x$ for which the function is defined .
  Remember that all denominators should not equal to 0 and what is inside the **even** nth root should be non-negative .
  Now let us find the domain of the following relations or functions :

  1. For $x = -2$ : The domain $= \{-2\}$
  2. For $y = 3\,;\, y = 2x+5\,;\, y = x^2-4\,;\, y = |x-2|\,;\, y = \sqrt[3]{x+4}\,;\, y = \dfrac{1}{x^2+9}$ and $y = \sqrt{x^2+1}$ : The domain$= \Re = (-\infty, \infty)$ .
  3. For $y = \dfrac{1}{x^2-2x-3}$ : We have $x^2-2x-3 \neq 0 \Rightarrow (x-3)(x+1) \neq 0 \Longrightarrow x \neq 3, -1$ . Therfore the domain $= (-\infty,-1) \cup (-1,3) \cup (3,\infty)$ .
  4. For $y = \sqrt{x^2-5x+4}$ : We have $x^2-5x+4 = (x-4)(x-1) \geq 0 \Longrightarrow$ ( by sign test ) $\boxed{-\infty < x \leq 1}$ or $\boxed{4 \leq x < \infty}$ . So the domain $= (-\infty, 1] \cup [4, \infty)$ .
  5. For $y = \dfrac{1}{\sqrt{9-x^2}}$ : Here $9-x^2 > 0 \Longrightarrow x^2 < 9 \Longrightarrow |x| < 3 \Longrightarrow -3 < x < 3$ . So the domain $= (-3,3)$ .
  6. For $y = \dfrac{1}{\sqrt[3]{x^2-x-2}}$ : Here $x^2-x-2$ can be negative but **not** 0 , i . e . , $x^2-x-2 = (x-2)(x+1) \neq 0 \Longrightarrow x \neq 2, -1 \Longrightarrow$ domain $= (-\infty,-1) \cup (-1,2) \cup (2,\infty)$
  7. For $y = \sqrt[4]{3-|x-2|}$ : In this case $3-|x-2| \geq 0 \Longrightarrow |x-2| \leq 3$ $\Longrightarrow -3 \leq x-2 \leq 3 \Longrightarrow -1 \leq x \leq 5 \Longrightarrow$ domain $= [-1,5]$ .
  8. Notice that the domain of $f(x) = \sqrt{\dfrac{x+1}{x-3}}$ is not the same as the domain of $g(x) = \dfrac{\sqrt{x+1}}{\sqrt{x-3}}$ . The reason for this is that in $f(x)$ : $\dfrac{x+1}{x-3} \geq 0$ while in $g(x)$ : $(x+1) \geq 0$ and $(x-3) > 0$ . In fact $D_f = (-\infty, -1] \cup (3,\infty)$ while $D_g = (3, \infty)$ .

- Second The **Range** : The range of $y = f(x)$ is the set of all possible real values of $y$ .
  We have to remember for example that : $|x| \geq 0$ ; $x^2 \geq 0$ ; $x^4 \geq 0$ for any $x \in \Re$ and $\sqrt{x} \geq 0$ ; $\sqrt[4]{x} \geq 0$ for any $x \geq 0$ . This implies that $|x|-2 \geq -2$ ; $x^2+5 \geq 5$ ; $x^4-8 \geq -8$ and $\sqrt{x}-3 \geq -3$ .

  Similarly $-|x| \leq 0$ ; $-x^2 \leq 0$ ; $-(x+2)^4 \leq 0$ for any $x \in \Re$ and $-\sqrt{x} \leq 0$ ; $-\sqrt[4]{x} \leq 0$ for any $x \geq 0$ . This implies that $-|x|+4 \leq 4$ ; $3-x^2 \leq 3$ ; $-\sqrt{x}-1 \leq -1$ etc ..

  In addition to this sometimes you need to solve $x$ in terms of $y$ to find the range of a relation or function.



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Next let us find the range of the following :

1. For $y = -3$ : The range $= \{-3\}$ .
2. For $x = 3$ ; $y = x - 2$ ; $y = x^3 + 1$ and $y = \sqrt[5]{x+4}$ : The range $= (-\infty, \infty)$ .
3. For $y = \sqrt{x+2} - 2$ : The range $= [-2, \infty)$ , because $\sqrt{x+2} \geq 0$ .
4. For $y = 3 - \sqrt[4]{x+1}$ : The range $= (-\infty, 3]$ , because $-\sqrt[4]{x+1} \leq 0$ .
5. For $y = |1 - 2x| - 4$ : The range $= [-4, \infty)$ , because $|1 - 2x| \geq 0$ .
6. For $y = x^2 - 6x$ : By completing to a perfect square $y = (x^2 - 6x + 9) - 9 = (x-3)^2 - 9 \geq -9$ , so the range $= [-9, \infty)$ .
7. For $y = \dfrac{1}{x-1}$ : Notice that $x = \dfrac{1+y}{y}$ , so $y \neq 0$ and then the range $= (-\infty, 0) \cup (0, \infty)$ .
8. For $y = \sqrt{x^2+4}$ : Notice that $\boxed{y \geq 0}$ and $x = \pm\sqrt{y^2 - 4}$ , so $y^2 - 4 \geq 0 \Longrightarrow y^2 \geq 4$ $\Longrightarrow |y| \geq 2 \Longrightarrow y \leq -2$ or $y \geq 2$ but $y \geq 0$ . Therefore the range $=$ $\boxed{(-\infty, -2] \cup [2, \infty)} \cap [0, \infty) = [2, \infty)$ .
9. For $y = \sqrt{9 - x^2}$ : Notice that $\boxed{y \geq 0}$ and $x = \pm\sqrt{9 - y^2}$ , so $9 - y^2 \geq 0 \Longrightarrow y^2 \leq 9$ $\Longrightarrow |y| \leq 3 \Longrightarrow -3 \leq y \leq 3$ but $y \geq 0 \Longrightarrow$ the range $= [0, 3]$ . We can also find the range of this function by noticing that the graph is the upper half of a circle of ceter $(0, 0)$ , radius 3 .
10. For $y = -\sqrt{x^2 - 16}$ : Notice that $\boxed{y \leq 0}$ and $x = \pm\sqrt{y^2 + 16}$ . The only restriction on $y$ is that $y \leq 0$ , so the range $= (-\infty, 0]$ .

**Exercise** :- Find the domain and range of the following :

1) $y = -|x - 8| + 8$
2) $y = \sqrt{x-2} - 3$
3) $x^2 - 4x + y^2 + 8y = 0$
4) $y = \sqrt{|x| - 4}$
5) $y = \dfrac{2x+1}{x-3}$
6) $y = \sqrt{\dfrac{x-5}{x+1}}$
7 ) $y = -\sqrt{x^2 - 16}$
8 ) $y = x^2 - x$
9 ) $y = \sqrt{x^2 - 6x + 9}$
10) $y = \sqrt{x^2 + 2x}$
11) $y = \dfrac{-1}{x^2 + 9}$
12) $y = \dfrac{\sqrt{x-5}}{\sqrt{x+1}}$

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