

Name : _____ ID. # : _____ SER. # : _____

• Be clean and show your work

1. Find the vertical asymptotes and the x-intercepts of the graph of $y = -3 \tan(2x + \frac{\pi}{2})$ over the interval $[-\pi, \frac{3\pi}{2}]$. (no need to graph) (3 pts)

Ans: The graph has vertical asymptotes when $2x + \frac{\pi}{2} = (2n + 1)\frac{\pi}{2} \implies 2x = (2n)\frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{2} = n\pi \implies x = n\frac{\pi}{2}, n \in Z$. So over the interval $[-\pi, \frac{3\pi}{2}]$: if $n = -2$, $x = -\pi$; if $n = -1$, $x = -\frac{\pi}{2}$; if $n = 0$, $x = 0$; if $n = 1$, $x = \frac{\pi}{2}$; if $n = 2$, $x = \pi$; if $n = 3$, $x = \frac{3\pi}{2}$

While the graph has x-intercepts when $2x + \frac{\pi}{2} = n\pi \implies 2x = n\pi - \frac{\pi}{2} = (2n - 1)\frac{\pi}{2} \implies x = (2n - 1)\frac{\pi}{4}, n \in Z$. So over the interval $[-\pi, \frac{3\pi}{2}]$: if $n = -1$, $x = -\frac{3\pi}{4}$;

if $n = 0$, $x = -\frac{\pi}{4}$; if $n = 1$, $x = \frac{\pi}{4}$; if $n = 2$, $x = \frac{3\pi}{4}$; if $n = 3$, $x = \frac{5\pi}{4}$.

Therefore the x-intercepts are $(-\frac{3\pi}{4}, 0), (-\frac{\pi}{4}, 0), (\frac{\pi}{4}, 0), (\frac{3\pi}{4}, 0)$, and $(\frac{5\pi}{4}, 0)$.

2. If $\sin \alpha = -\frac{4}{5}$, $\pi < \alpha < \frac{3\pi}{2}$ and $\cos \beta = -\frac{12}{13}$, $\frac{\pi}{2} < \beta < \pi$, find

$$\tan(\pi + \alpha + \beta), \quad \cos(\beta + \frac{\pi}{4}), \quad \cos(\alpha - \beta + \frac{\pi}{2}) \quad (4 \text{ pts})$$

Ans: α lies in Q III, $\sin \alpha = -\frac{4}{5} = \frac{y}{r} \implies y = -4, r = 5, x = -\sqrt{25 - 16} = -3$

$\implies \cos \alpha = -\frac{3}{5}, \tan \alpha = \frac{4}{3}$ and β lies in Q II, $\cos \beta = -\frac{12}{13} = \frac{x}{r} \implies x = -12, r = 13,$

$y = \sqrt{169 - 144} = 5 \implies \sin \beta = \frac{5}{13}, \tan \beta = -\frac{5}{12}$. So

$$(i) \tan(\pi + \alpha + \beta) = \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \cdot \tan \beta} = \frac{\frac{4}{3} + (-\frac{5}{12})}{1 - (\frac{4}{3})(-\frac{5}{12})} \cdot \frac{36}{36} = \frac{48 - 15}{36 + 20} = \frac{33}{56}$$

$$(ii) \cos(\beta + \frac{\pi}{4}) = \cos \beta \cos \frac{\pi}{4} - \sin \beta \sin \frac{\pi}{4} = (-\frac{12}{13}) (\frac{\sqrt{2}}{2}) - (\frac{5}{13}) (\frac{\sqrt{2}}{2}) = -\frac{17\sqrt{2}}{26}$$

$$(iii) \cos(\alpha - \beta + \frac{\pi}{2}) = \cos[\frac{\pi}{2} - (\beta - \alpha)] = \sin(\beta - \alpha) = \sin \beta \cos \alpha - \cos \beta \sin \alpha = (\frac{5}{13}) (-\frac{3}{5}) - (-\frac{12}{13}) (-\frac{4}{5}) = -\frac{63}{65}$$

3. Verify the identity $1 - \sec^4 x = -\frac{\sin^2 x (1 + \cos^2 x)}{\cos^4 x}$ (3 pts)

Ans: L.H.S. = $(1 - \sec^2 x)(1 + \sec^2 x) = (-\tan^2 x)(1 + \sec^2 x) = -\frac{\sin^2 x}{\cos^2 x} \cdot (1 + \frac{1}{\cos^2 x})$

$$= -\frac{\sin^2 x}{\cos^2 x} \cdot \frac{\cos^2 x + 1}{\cos^2 x} = -\frac{\sin^2 x (1 + \cos^2 x)}{\cos^4 x} = \text{R.H.S.}$$

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1. Find the domain and the range of the function $y = -2 \sec(3x + \frac{\pi}{4}) + 5$ (3 pts)

Ans: For domain: $3x + \frac{\pi}{4} \neq (2n+1)\frac{\pi}{2} \implies 3x \neq n\pi + \frac{\pi}{2} - \frac{\pi}{4} = n\pi + \frac{\pi}{4} = (4n+1)\frac{\pi}{4} \implies x \neq (4n+1)\frac{\pi}{12}, n \in Z$. Therefore the domain = $(-\infty, \infty) - \{(4n+1)\frac{\pi}{12}, n \in Z\}$.

For range: $a = -2, d = 5$, the range = $(-\infty, -|a| + d] \cup [|a| + d, \infty) = (-\infty, -2 + 5] \cup [2 + 5, \infty) = (-\infty, 3] \cup [7, \infty)$

2. Find the values of the following: $\cos 165^\circ$, $\csc 15^\circ$, $\frac{\tan(\frac{\pi}{7} + \frac{2\pi}{3}) - \tan \frac{\pi}{7}}{1 + \tan(\frac{\pi}{7} + \frac{2\pi}{3}) \cdot \tan \frac{\pi}{7}}$ (4 pts)

Ans: (i) $\cos 165^\circ = \cos(45^\circ + 120^\circ) = \cos 45^\circ \cos 120^\circ - \sin 45^\circ \sin 120^\circ$
 $= (\frac{\sqrt{2}}{2})(-\frac{1}{2}) - (\frac{\sqrt{2}}{2})(\frac{\sqrt{3}}{2}) = -\frac{\sqrt{2} + \sqrt{6}}{4}$

(ii) $\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ = (\frac{\sqrt{2}}{2})(\frac{\sqrt{3}}{2}) - (\frac{\sqrt{2}}{2})(\frac{1}{2})$
 $= \frac{\sqrt{6} - \sqrt{2}}{4} \implies \csc 15^\circ = \frac{1}{\sin 15^\circ} = \frac{4}{\sqrt{6} - \sqrt{2}} \cdot \frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} + \sqrt{2}} = \frac{4(\sqrt{6} + \sqrt{2})}{6 - 2} = \sqrt{6} + \sqrt{2}$

(iii) $\frac{\tan(\frac{\pi}{7} + \frac{2\pi}{3}) - \tan \frac{\pi}{7}}{1 + \tan(\frac{\pi}{7} + \frac{2\pi}{3}) \cdot \tan \frac{\pi}{7}} = \tan[(\frac{\pi}{7} + \frac{2\pi}{3}) - \frac{\pi}{7}] = \tan \frac{2\pi}{3} = -\tan \frac{\pi}{3} = -\sqrt{3}$

3. Verify the identity $\frac{2 \cos x}{\sin x - 1} + \frac{\cos x}{1 - \sin x} = -\frac{1 + \sin x}{\cos x}$ (3 pts)

Ans: L.H.S. = $-\frac{2 \cos x}{1 - \sin x} + \frac{\cos x}{1 - \sin x} = -\frac{\cos x}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x} = -\frac{\cos x (1 + \sin x)}{1 - \sin^2 x}$
 $= -\frac{\cos x (1 + \sin x)}{\cos^2 x} = -\frac{1 + \sin x}{\cos x} = \text{R.H.S.}$

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1. Find the vertical asymptotes and the x-intercepts of the graph of $y = 4 \cot(2x + \frac{\pi}{2})$ over the interval $[-\pi, \frac{3\pi}{2}]$. (no need to graph) (3 pts)

Ans: The graph has vertical asymptotes when $2x + \frac{\pi}{2} = n\pi \implies 2x = n\pi - \frac{\pi}{2} = (2n-1)\frac{\pi}{2} \implies x = (2n-1)\frac{\pi}{4}, n \in Z$. So over the interval $[-\pi, \frac{3\pi}{2}]$: if $n = -1$, $x = -\frac{3\pi}{4}$;

if $n = 0$, $x = -\frac{\pi}{4}$; if $n = 1$, $x = \frac{\pi}{4}$; if $n = 2$, $x = \frac{3\pi}{4}$; if $n = 3$, $x = \frac{5\pi}{4}$.

While the graph has x-intercepts when $2x + \frac{\pi}{2} = (2n+1)\frac{\pi}{2} \implies 2x = (2n)\frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{2} = n\pi \implies x = n\frac{\pi}{2}, n \in Z$. So over the interval $[-\pi, \frac{3\pi}{2}]$: if $n = -2$, $x = -\pi$; if $n = -1$, $x = -\frac{\pi}{2}$; if $n = 0$, $x = 0$; if $n = 1$, $x = \frac{\pi}{2}$; if $n = 2$, $x = \pi$; if $n = 3$, $x = \frac{3\pi}{2}$.

Therefore the x-intercepts are $(-\pi, 0), (-\frac{\pi}{2}, 0), (0, 0), (\frac{\pi}{2}, 0), (\pi, 0)$, and $(\frac{3\pi}{2}, 0)$.

2. If $\sin \alpha = \frac{3}{5}$, $\frac{\pi}{2} < \alpha < \pi$ and $\cos \beta = \frac{5}{13}$, $\frac{3\pi}{2} < \beta < 2\pi$, find

$$\sin(\alpha + \frac{5\pi}{3}), \quad \cos(\frac{\pi}{2} + \alpha - \beta), \quad \frac{\tan(\frac{\alpha}{2} + \beta) + \tan(\frac{\alpha}{2} - \beta)}{1 - \tan(\frac{\alpha}{2} + \beta) \cdot \tan(\frac{\alpha}{2} - \beta)} \quad (4 \text{ pts})$$

Ans: α lies in Q II, $\sin \alpha = \frac{3}{5} = \frac{y}{r} \implies y = 3, r = 5, x = -\sqrt{25-9} = -4 \implies \cos \alpha = -\frac{4}{5}$,

$\tan \alpha = -\frac{3}{4}$ and β lies in Q IV, $\cos \beta = \frac{5}{13} = \frac{x}{r} \implies x = 5, r = 13, y = -\sqrt{169-25} = -12$

$\implies \sin \beta = -\frac{12}{13}$. So

$$(i) \sin(\alpha + \frac{5\pi}{3}) = \sin \alpha \cos \frac{5\pi}{3} + \cos \alpha \sin \frac{5\pi}{3} = (\frac{3}{5})(\frac{1}{2}) + (-\frac{4}{5})(-\frac{\sqrt{3}}{2}) = \frac{3+4\sqrt{3}}{10}$$

$$(ii) \cos(\frac{\pi}{2} + \alpha - \beta) = \cos[\frac{\pi}{2} - (\beta - \alpha)] = \sin(\beta - \alpha) = \sin \beta \cos \alpha - \cos \beta \sin \alpha$$

$$= (-\frac{12}{13})(-\frac{4}{5}) - (\frac{5}{13})(\frac{3}{5}) = \frac{48-15}{65} = \frac{33}{65}$$

$$(iii) \frac{\tan(\frac{\alpha}{2} + \beta) + \tan(\frac{\alpha}{2} - \beta)}{1 - \tan(\frac{\alpha}{2} + \beta) \cdot \tan(\frac{\alpha}{2} - \beta)} = \tan[(\frac{\alpha}{2} + \beta) + (\frac{\alpha}{2} - \beta)] = \tan \alpha = -\frac{3}{4}$$

3. Verify the identity $\sqrt{\frac{1+\cos x}{1-\cos x}} = \frac{1+\cos x}{\sin(-x)}$, $\sin x < 0$ (3 pts)

$$\text{Ans: L.H.S.} = \sqrt{\frac{1+\cos x}{1-\cos x}} \cdot \frac{1+\cos x}{1+\cos x} = \sqrt{\frac{(1+\cos x)^2}{1-\cos^2 x}} = \sqrt{\frac{(1+\cos x)^2}{\sin^2 x}} = \frac{|1+\cos x|}{|\sin x|}$$

$$= \frac{1+\cos x}{-\sin x} = \frac{1+\cos x}{\sin(-x)} = \text{R.H.S.} \quad (\text{because } 1+\cos x > 0 \text{ and } \sin x < 0 \text{ (given)})$$