MATH 002 – Class Test **Semester II – 2004-2005 (042)**

• Time Allowed : 75 Minutes + **• No Calculators** **• Show your work**

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1.(a) Express $\tan x$ in terms of $\csc x$, where $\frac{3\pi}{2} < x < 2\pi$. (6 pts)

Ans : $x \in$ Q IV, so $\tan x < 0$ and

$$\tan x = \frac{1}{\cot x} = \frac{1}{-\sqrt{\csc^2 x - 1}} = -\frac{1}{\sqrt{\csc^2 x - 1}}$$

(b) Find the value of the following: (11 pts)

(i) $(\cos 15^\circ + \sin 15^\circ)^2 + 1 - 2\cos^2 75^\circ$ (ii) $\sin^{-1}(\sin\frac{3\pi}{2}) + \cos^{-1}(\cos\frac{7\pi}{4}) + \sec^{-1}(-2)$

Ans : (i) $= \cos^2 15^\circ + 2\sin 15^\circ \cos 15^\circ + \sin^2 15^\circ - (2\cos^2 75^\circ - 1) = 1 + \sin 2(15^\circ) - \cos 2(75^\circ)$

$$= 1 + \sin 30^\circ - \cos 150^\circ = 1 + \frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{3+\sqrt{3}}{2}$$

(ii) $= \sin^{-1}(-1) + \cos^{-1}(\frac{\sqrt{2}}{2}) + (\pi - \sec^{-1} 2) = -\frac{\pi}{2} + \frac{\pi}{4} + (\pi - \frac{\pi}{3}) = \frac{3\pi}{4} - \frac{\pi}{3} = \frac{5\pi}{12}$

(c) Let $\vec{u}$ be a **unit** vector with direction angle 120°. Find the magnitude and direction angle of the vector $\vec{w} = -2\vec{u} + 2\vec{\imath} - 2\sqrt{3}\vec{\jmath}$. (11 pts)

Ans : $||\vec{u}|| = 1$, so $\vec{u} = < \cos 120^\circ , \sin 120^\circ > = < -\frac{1}{2}, \frac{\sqrt{3}}{2} >$. Now

$\vec{w} = -2 < -\frac{1}{2}, \frac{\sqrt{3}}{2} > + < 2, -2\sqrt{3} > = < 1+2, -\sqrt{3} - 2\sqrt{3} > = < 3, -3\sqrt{3} >$. Therefore

$||\vec{w}|| = \sqrt{x^2 + y^2} = \sqrt{9 + 27} = 6$ and $\cos\theta = \frac{x}{||\vec{w}||} = \frac{3}{6} = \frac{1}{2}$, $\sin\theta = \frac{y}{||\vec{w}||} = -\frac{3\sqrt{3}}{6}$

$= -\frac{\sqrt{3}}{2} \implies \theta = 360^\circ - 60^\circ = 300^\circ$.

2.(a) Verify the identity: $\dfrac{\cos 2x \cos 4x + \sin 2x \sin 4x}{\sin 3x \cos 5x - \cos 3x \sin 5x} = \dfrac{\tan^2 x - 1}{2\tan x}$ (9 pts)

Ans : L.H.S. $= \dfrac{\cos(2x-4x)}{\sin(3x-5x)} = \dfrac{\cos(-2x)}{\sin(-2x)} = \cot(-2x) = -\dfrac{1}{\tan x} = -\dfrac{1-\tan^2 x}{2\tan x}$

$= \dfrac{\tan^2 x - 1}{2\tan x} =$ R.H.S.

(b) Solve the following equations: (14 pts)

(i) $\tan^2 x - \tan x + \sqrt{3}\tan x - \sqrt{3} = 0,$ $0 \le x < 2\pi$; (ii) $\cos^{-1}(2x) + \cos^{-1}(\frac{3}{5}) = \frac{\pi}{2}$

Ans : (i) $= \tan x(\tan x - 1) + \sqrt{3}(\tan x - 1) = 0 \implies (\tan x - 1)(\tan x + \sqrt{3}) = 0 \implies$ $\tan x = 1$ or $\tan x = -\sqrt{3}$. When $\tan x = 1$, then $x = \frac{\pi}{4}$ or $x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$ and when $\tan x = -\sqrt{3}$, then $x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ or $x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$. Therefore S.S. $= \{\frac{\pi}{4}, \frac{2\pi}{3}, \frac{5\pi}{4}, \frac{5\pi}{3}\}$

(ii) $\cos^{-1}(2x) = \frac{\pi}{2} - \cos^{-1}\frac{3}{5} \implies 2x = \cos[\frac{\pi}{2} - \cos^{-1}\frac{3}{5}] = \sin(\cos^{-1}\frac{3}{5}) = \sin\alpha$, where $\alpha = \cos^{-1}\frac{3}{5} \in$ Q I $\implies 2x = \sin\alpha = \frac{4}{5} \implies x = \frac{2}{5}$

(c) Find the **equation** of the ellipse with foci $F_1(-1,4)$, $F_2(-1,8)$ and passing through the point $P(-3,6)$. Also find the vertices of the ellipse. (9 pts)

Ans : Center $= C(\frac{-1-1}{2}, \frac{4+8}{2}) = C(-1,6)$, major axis $\parallel$ y–axis.

$d(F_1,F_2) = 2c \implies 8-4 = 4 = 2c \implies \boxed{c=2}$ and $d(P,F_1) + d(P,F_2) = 2a$

$\implies \sqrt{4+4} + \sqrt{4+4} = 2\sqrt{8} = 2a \implies \boxed{a = \sqrt{8} = 2\sqrt{2}}$, $b^2 = a^2 - c^2 = 8 - 4 = 4.$

So the equation is $\dfrac{(x+1)^2}{4} + \dfrac{(y-6)^2}{8} = 1$ and the vertices are

$V_1(h, a+k) = V_1(-1, 2\sqrt{2}+6)$, $V_2(h, -a+k) = V_2(-1, -2\sqrt{2}+6)$

3. Complete the following (show your work and simplify your answer): (40 pts)

- If $\sin\frac{\theta}{2} = \frac{3}{5}$, $\frac{3\pi}{2} < \theta < 2\pi$, then $\sin\theta = \cdots\cdots$

  <u>Ans</u> : $\frac{3\pi}{4} < \frac{\theta}{2} < \pi \implies \frac{\theta}{2} \in$ Q II , $\cos\frac{\theta}{2} = -\frac{4}{5}$. Then $\sin\theta = 2\sin\theta\cos\theta = 2(\frac{3}{5})(-\frac{4}{5}) = -\frac{24}{25}$

- The **eccentricity** e of the ellipse $4x^2 + 9y^2 = 36$ is $\cdots\cdots\cdots$

  <u>Ans</u> : $\frac{x^2}{9} + \frac{y^2}{4} = 1 \implies a = 3\,, b = 2$ and $c^2 = a^2 - b^2 = 9 - 4 = 5 \implies c = \sqrt{5}$

  $\implies e = \frac{c}{a} = \frac{\sqrt{5}}{3}$

- The **minimum** value of the function $y = 3\sin x + 4\cos x + 7$ is equal to $\cdots\cdots\cdots$

  <u>Ans</u> : $3\sin x + 4\cos x = k\sin(x+\alpha)$, where $k = \sqrt{a^2 + b^2} = \sqrt{9+16} = 5$.

  Then $y = 5\sin(x+\alpha) + 7 \implies$ the range $= [-5+7, 5+7] = [2, 12]$

  $\implies$ the minimum $= 2$

- If $\frac{\pi}{4} < x < \frac{\pi}{2}$, then $\sqrt{\frac{8}{1+\cos 4x}} = \cdots\cdots\cdots$

  <u>Ans</u> : $\frac{\pi}{2} < 2x < \pi \implies 2x \in$ Q II. So $\sqrt{\frac{8}{1+\cos 4x}} = 2\left(\sqrt{\frac{2}{1+\cos 4x}}\right) = 2|\sec 2x| = -2\sec 2x$

- The **range** of the function $y = -2\cos^{-1}(3x+1) + 5\pi$ is $\cdots\cdots\cdots$

  <u>Ans</u> : $0 \le \cos^{-1}(3x+1) \le \pi \implies 0 \ge -2\cos^{-1}(3x+1) \ge -2\pi \implies 0 + 5\pi \ge y \ge -2\pi + 5\pi$

  $\implies 5\pi \ge y \ge 3\pi \implies$ the range $= [3\pi, 5\pi]$

- The **focus** of the parabola $x^2 - 2x + 3y = 0$ is $\cdots\cdots\cdots$

  Ans : $x^2 - 2x + 1 = -3y + 1 \implies (x-1)^2 - 3(y - \frac{1}{3})$

  This is a parabola opens downward with vertex $V(1, \frac{1}{3})$ and $4p = -3 \implies p = -\frac{3}{4}$

  $\implies$ the focus $= F(h, p+k) = F(1, -\frac{3}{4} + \frac{1}{3}) = F(1, -\frac{5}{12})$

- The **sum** of all soltions of the equation $\sin x \cos x = \frac{1}{2}$, $0 \leq x < 2\pi$ is $\cdots\cdots\cdots$

  Ans : $2\sin x \cos x = 1 \implies \sin 2x = 1,\ 0 \leq 2x < 4\pi \implies 2x = \frac{\pi}{2}$ or $\frac{5\pi}{2}$

  $\implies x = \frac{\pi}{4}$ or $\frac{5\pi}{4} \implies$ the sum $= \frac{\pi}{4} + \frac{5\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$

- The **equation** of the **asymptote** with **negative slope** of the hyperbola $4x^2 - 5y^2 + 20 = 0$ is $\cdots\cdots\cdots$

  Ans : $4x^2 - 5y^2 = -20 \implies \frac{y^2}{4} - \frac{x^2}{5} = 1 \implies a = 2\,,\ b = \sqrt{5}.$

  The asymptotes are $y = \pm\frac{a}{b}x = \pm\frac{2}{\sqrt{5}}x \implies$ the one with negative slope is $y = -\frac{2}{\sqrt{5}}x.$

- $\dfrac{\tan 50^\circ + \tan 10^\circ}{1 - \cot 40^\circ \cot 80^\circ} = \cdots\cdots\cdots$

  Ans : $= \dfrac{\tan 50^\circ + \tan 10^\circ}{1 - \tan 50^\circ \tan 10^\circ} = \tan(50^\circ + 10^\circ) = \tan 60^\circ = \sqrt{3}$

- If $\vec{u} =< 3\cos 25^\circ\ , 3\sin 25^\circ >$, $\vec{v} =< 2\cos 35^\circ\ , -2\sin 35^\circ >$, then $\vec{u} \cdot \vec{v} = \cdots\cdots$

  Ans : $\vec{u} \cdot \vec{v} = (3\cos 25^\circ)(2\cos 35^\circ) + (3\sin 25^\circ)(-2\sin 35^\circ) = 6\cos 25^\circ \cos 35^\circ - 6\sin 25^\circ \sin 35^\circ$

  $= 6(\cos 25^\circ \cos 35^\circ - \sin 25^\circ \sin 35^\circ) = 6\cos(25^\circ + 35^\circ) = 6\cos 60^\circ = 6(\frac{1}{2}) = 3$