Math 002 — Vectors (Done By: A. Al-Shallali)

• a vector $\vec{u}$ is a directed line segment AB. We write $\vec{u} = \overrightarrow{AB}$. Point A is called the initial point (tail) and B is the terminal point (head) of $\vec{u}$. If $\vec{u}$ is directed from the origin $(0,0)$ to the point (a,b), then the vector $\vec{u}$ is written as $\vec{u} =< a,b >$. The number a is called the horizontal (x) - component and b is called the vertical (y) - component of $\vec{u}$.
If $\vec{u}$ is directed from the point (a_1,b_1) to the point (a_2,b_2), then we write $\vec{u} =< a_2 - a_1, b_2 - b_2 >$.
The vector $\vec{u} =< a,b >$ has **magnitude** (length) $= r = ||\vec{u}|| = \sqrt{a^2+b^2} \geq 0$ and **direction angle** θ which is the smallest positive angle measured in the counterclockwise direction from the positive $x-$axis to the vector $\vec{u}$, so $0° \leq \theta < 360°$.
It can be shown that for any vector $\vec{u} = < x,y >$: $x = ||\vec{u}|| \cos\theta$, $y = ||\vec{u}|| \sin\theta$ and $\tan\theta = \frac{y}{x}$
So $\vec{u} = < x,y > = < ||\vec{u}|| \cos\theta, ||\vec{u}|| \sin\theta >$, do not forget again that $||\vec{u}|| \geq 0$ and $0° \leq \theta < 360°$.

Example : Let $\vec{u}$ be a vector of magnitude 4 and direction angle 210°. If the vector $\vec{v} = \overrightarrow{AB}$ is **equivalent** to the vector $\vec{u}$ and has initial point A$(-2,3)$, find its terminal point B.

Ans : $\vec{v} = \overrightarrow{AB} = \vec{u} = < 4\cos 210^{\circ}, 4\sin 210^{\circ} > = < -4\cos 30^{\circ}, -4\sin 30^{\circ} > = < -4(\sqrt{3}/2), -4(1/2) >$
$= < -2\sqrt{3}, -2 >$. Assume the terminal point = B(a,b) $\Longrightarrow$ $\vec{v} = < -2\sqrt{3}, -2 > = < a+2, b-3 >$
$\Longrightarrow$ $a+2 = -2\sqrt{3}$, $b-3 = -2$ $\Longrightarrow$ $a = -2-2\sqrt{3}$, $b = 1$ $\Longrightarrow$ terminal point = B$(-2-2\sqrt{3}, 1)$.

Example : Find the magnitude and direction angle of the following vectors :

1. $\vec{u} = < 3\cos 20°, 3\sin(-20)° >$: We have $\vec{u} = < 3\cos(-20)°, 3\sin(-20)° >$
$= < 3\cos(-20+360)°, 3\sin(-20+360)° > = < 3\cos 340°, 3\sin 340° >$. Then the magnitude $= ||\vec{u}|| = 3$ and $\theta = 340°$.

2. $\vec{w} =< -2\sqrt{3}, -2 >$: magnitude $= ||\vec{w}|| = \sqrt{12+4} = \sqrt{16} = 4$, $\sin\theta = \frac{y}{r} = \frac{-2}{4} = -\frac{1}{2}$ and
$\cos\theta = \frac{x}{r} = \frac{-2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}$, so θ lies in the third quadrant and equals to $180° + 30° = 210°$.

3. $\vec{v} =< 3, -4 >$: magnitude $= ||\vec{v}|| = \sqrt{9+16} = 5$, $\cos\theta = \frac{3}{5}$, and $\sin\theta = -\frac{4}{5}$,
$\Longrightarrow$ $\theta' = \cos^{-1}\frac{3}{5}$ or $\sin^{-1}\frac{4}{5}$, where θ lies in the fourth quadrant. Therefore
$\theta = 360^{\circ} - \cos^{-1}\frac{3}{5} = 360^{\circ} - \sin^{-1}\frac{4}{5}$ (use a calculator to find the exact value of θ)

It is advisable to sketch the vector to help you in finding the correct value of θ .

• Two vectors have the **same direction** if they have the **same direction angle** .
Two vectors with direction angles θ_1 and θ_2 have **opposite directions** if $|\theta_2 - \theta_1| = 180°$.

• For any vectors $\vec{u} = < a,b >$, $\vec{v} = < c,d >$ and $k \in \Re$ we have
(i) $\vec{u} + \vec{v} = < a,b > + < c,d > = < a+c, b+d >$ (ii) $k\vec{u} = < ka, kb >$

• **Def.** : A vector $\vec{u}$ is called a **unit** vector if $||\vec{u}|| = 1$.
Notice that any **unit** vector has the form $< \cos\theta, \sin\theta >$, where θ is its direction angle.
For example $< 1,0 >$; $< 0,1 >$; $< -1,0 >$; $< 0,-1 >$; $< -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} >$; $< -\frac{\sqrt{3}}{2}, -\frac{\sqrt{1}}{2} >$; and $< -\frac{3}{5}, -\frac{4}{5} >$ all are **unit** vectors . In fact any vector directed from the origin to any point on the edge of the unit circle is always a unit vector.
The unit vectors $\vec{\imath}$ and $\vec{\jmath}$ respectively are $< 1,0 >$ and $< 0,1 >$. Therefore for any vector $\vec{u}$:
$\vec{u} = < x,y > = < x,0 > + < 0,y > = x < 1,0 > + y < 0,1 > = x\vec{\imath} + y\vec{\jmath} = ||\vec{u}|| \cos\theta\vec{\imath} + ||\vec{u}|| \sin\theta\vec{\jmath}$.

• Let $\vec{u}$ be a vector with magnitude r and direction angle θ. If $t > 0$ and $k \in \Re - \{0\}$, then

1. For $k > 0$: $\vec{v} = k\vec{u}$ is a vector with magnitude kr and has the <u>**same**</u> direction of $\vec{u}$.

2. For $k < 0$: $\vec{v} = k\vec{u}$ is a vector with magnitude $|k|r$ and has an <u>**opposite**</u> direction of $\vec{u}$.

   In this case, the direction angle of $k\vec{u}$ is $|180^{\circ} - \theta|$.

3. $\frac{\vec{u}}{||\vec{u}||} = \frac{1}{||\vec{u}||} < ||\vec{u}|| \cos\theta, ||\vec{u}|| \sin\theta > = < \cos\theta, \sin\theta >$ is a **unit** vector in the **same** direction of $\vec{u}$.

4. $-\frac{\vec{u}}{||\vec{u}||} = \frac{-1}{||\vec{u}||} < ||\vec{u}|| \cos\theta, ||\vec{u}|| \sin\theta > = < -\cos\theta, -\sin\theta >$ is a **unit** vector in an **opposite** direction of $\vec{u}$.

5. $\frac{t}{||\vec{u}||}\ \vec{u}$ is a vector of magnitude t and has the **same** direction of $\vec{u}$.

6. $-\frac{t}{||\vec{u}||}\ \vec{u}$ is a vector of magnitude t and has an **opposite** direction of $\vec{u}$.

• The **dot (inner) product** of two vectors $\vec{u} = < a, b >$ and $\vec{v} = < c, d >$ is given by : $\vec{u} \cdot \vec{v} = < a, b > \cdot < c, d > = ac + bd$. So $\vec{u} \cdot \vec{v}$ is not a vector any more , it is a scalar , e . g . , if $\vec{u} = 3\vec{\imath} + 2\vec{\jmath}$ and $\vec{v} = < 1, -4 >$, then $3\vec{u} \cdot 2\vec{v} = < 9, 6 > \cdot < 2, -8 > = (9)(2) + (6)(-8) = -30$. Notice that $\vec{u}^2 = (\vec{u})(\vec{u})$ is a meaningless expression in vectors , instead , we have $\vec{u} \cdot \vec{u}$ and $||\vec{u}||^2$, where both are scalars and are equal to $x^2 + y^2$ if $\vec{u} = < x, y >$.

Example : If $\vec{u} =< 2\sin 80°, \cos 80° >$ and $\vec{v} =< \sin 80°, 2\cos 80° >$, then $\vec{u} \cdot \vec{v} = (2\sin 80°)(\sin 80°) + (\cos 80°)(2\cos 80°) = 2\sin^2 80° + 2\cos^2 80° = 2(\sin^2 80° + \cos^2 80°) = 2(1) = 2$.

• The angle α between two vectors $\vec{u}$ & $\vec{v}$: It is the smallest positive angle between them , i . e . , $0° \leq \alpha \leq 180°$. The angle α can be found by using : $\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos\alpha$.
Therfore $\alpha = \cos^{-1}\frac{\vec{u} \cdot \vec{v}}{||\vec{u}||\ ||\vec{v}||}$, where $0° \leq \alpha \leq 180°$.
Do not confuse between θ (direction angle of a vector) and α (angle between two vectors) .
Example : Find the angle between the given vectors :

1. $\vec{u} = < 3, -1 >$, $\vec{v} = < -6, 2 >$: We have
   $\vec{u} \cdot \vec{v} = -18 - 2 = -20$, $||\vec{u}|| = \sqrt{10}$, $||\vec{v}|| = \sqrt{40} = 2\sqrt{10}$, then the angle
   $\alpha = \cos^{-1}\frac{\vec{u} \cdot \vec{v}}{||\vec{u}||\ ||\vec{v}||} = \cos^{-1}\frac{-20}{(\sqrt{10})(2\sqrt{10})} = \cos^{-1}(-1) = 180°$.

2. $\vec{u} = < 5, 2 >$, $\vec{v} = < -4, 10 >$: We have
   $\vec{u} \cdot \vec{v} = -20 + 20 = 0$, then $\alpha = \cos^{-1}\frac{\vec{u} \cdot \vec{v}}{||\vec{u}||\ ||\vec{v}||} = \cos^{-1}0 = 90°$.

3. $\vec{u} = < -2, 2\sqrt{3} >$, $\vec{v} = < \sqrt{6}, -\sqrt{2} >$: We have
   $\vec{u} \cdot \vec{v} = -2\sqrt{6} - 2\sqrt{6} = -4\sqrt{6}$, $||\vec{u}|| = \sqrt{16} = 4$, $||\vec{v}|| = \sqrt{8} = 2\sqrt{2}$, then
   $\alpha = \cos^{-1}\frac{\vec{u} \cdot \vec{v}}{||\vec{u}||\ ||\vec{v}||} = \cos^{-1}\frac{-4\sqrt{6}}{(4)\ (2\sqrt{2})} = \cos^{-1}(-\frac{\sqrt{3}}{2}) = 180° - 30° = 150°$.

• Notice the following :

1. $\vec{u}$ and $\vec{v}$ are two non-zero orthogonal vectors ($\alpha = 90°$) iff $\vec{u} \cdot \vec{v} = 0$.

2. If $\vec{u} = < x, y >$, then $\vec{v} = < -y, x >$ and $\vec{w} = < y, -x >$ are both orthogonal to $\vec{u}$ because $\vec{u} \cdot \vec{v} = 0$ and $\vec{u} \cdot \vec{w} = 0$.

3. For any vector $\vec{u}$: $\vec{u} \cdot \vec{u} = ||\vec{u}||\ ||\vec{u}||\ \cos 0° = ||\vec{u}||^2$.

4. If $\vec{u}$ and $\vec{v}$ have the same direction , then $\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}||\ \cos 0° = ||\vec{u}||\ ||\vec{v}||$.

5. If $\vec{u}$ and $\vec{v}$ are opposite in direction , then $\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}||\ \cos 180° = -||\vec{u}||\ ||\vec{v}||$.

6. If $\vec{u} = < x, y >$, then $\pm\frac{1}{||\vec{u}||} < y, -x >$ are the unit vectors orthogonal to $\vec{u}$.

- Let $\vec{u}$ and $\vec{v}$ be two different non-zero vectors and α be the angle between them, then :

1. $||\vec{u} + \vec{v}||^2 = ||\vec{u}||^2 + 2\, \vec{u}\cdot\vec{v} + ||\vec{v}||^2$ (by using $||\vec{w}||^2 = \vec{w}\cdot\vec{w}$)
2. $||\vec{u} - \vec{v}||^2 = ||\vec{u}||^2 - 2\, \vec{u}\cdot\vec{v} + ||\vec{v}||^2$
3. $||\vec{u} + \vec{v}|| = ||\vec{u} - \vec{v}||$ if $\alpha = 90^{\circ}$. (because $\cos\alpha = 0 \implies \vec{u}\cdot\vec{v} = 0$)
4. $||\vec{u} + \vec{v}|| > ||\vec{u} - \vec{v}||$ if α is an acute angle. (because $\cos\alpha > 0$)
5. $||\vec{u} + \vec{v}|| < ||\vec{u} - \vec{v}||$ if α is an obtuse angle. (because $\cos\alpha < 0$)

Problems on Vectors

1. Find the **magnitude** and the **dirction angle** of the vectors : i) $< 2\sqrt{3}, -2 >$ ii) $< -4, -4 >$
   iii) $< 2\cos 50^\circ, 2\sin 50^\circ >$ iv) $< \sin 70^\circ, \cos 70^\circ >$ v) $< -2\sin 10^\circ, -2\cos 10^\circ >$
2. If $\vec{u} =< 2, -1 >$ and $\vec{v} =< 3, 4 >$, then find the magnitude of the vector $3\vec{u} - 2\vec{v} - 5\vec{\imath} - \vec{\jmath}$.
3. If $\vec{u} =< 1, -2 >$ and $\vec{v} =< \sqrt{3}, -3 >$, find the **dirction angle** of the vector $2\vec{u} - 2\vec{v} - 2\vec{\imath}$.
4. If $\vec{u}$ is a vector with **magnitude** 4 and **dirction angle** 330°, then find the x and y components of the vector $2\vec{u}$.
5. If $\vec{u}$ is a vector with **magnitude** 2 and **dirction angle** 120° and $\vec{v}$ is a vector with **magnitude** 4 and **dirction angle** 210° , find $\sqrt{3}\,\vec{u} - 2\vec{v}$.
6. Find the **magnitude** of the vector $\vec{u} =< \cos\alpha + \sin\alpha, \cos\alpha - \sin\alpha >$
7. Which one of the following vectors is a **unit** vector ?
   i) $< \frac{1}{2}, \frac{1}{2} >$ ii) $< \frac{1}{4}\cos 50^\circ, \frac{3}{4}\sin 50^\circ >$ iii) $< \cos 20^\circ, \cos 70^\circ >$
8. If $\vec{u} =< 3, -4 >$,find :
   (a) The **unit** vector in the **same** direction of $\vec{u}$.
   (b) The vector of **magnitude** 4 and has **opposite** direction of $\vec{u}$.
   (c) The vector(s) of magnitude 10 and **orthogonal** to $\vec{u}$.
9. If θ is the dirction angle of the vector $< -2, 3 >$, then find $\sin 2\theta$.
10. If α is the angle between the unit vectors $\vec{u}$ and $\vec{v}$ such that $\cos\alpha = -\frac{1}{9}$, then find $||\vec{u} + \vec{v}||$.
11. If $\vec{u}$ and $\vec{v}$ are two orthogonal vectors of magnitude 2 and 3, then find the magnitude of the vector $3\vec{u} - 2\vec{v}$.

Ans. : 1) i) 4 ,330° , ii) 4 $\sqrt{2}$, 225° ,iii) 2 ,50° , iv) 1 ,20° ,v) 2 ,$80^\circ + 180^\circ = 260^\circ$; 2) 13 ; 3) 150°

4) $x = 4\sqrt{3}$, $y = -4$; 5) $< 3\sqrt{3}, 7 >$; 6) $\sqrt{2}$; 7) The only unit vector is (iii)

8) a) $< \frac{3}{5}, -\frac{4}{5} >$, b) $< -\frac{12}{5}, \frac{16}{5} >$, c) $\pm < 8, 6 >$; 9) $-\frac{12}{13}$; 10) $\frac{4}{3}$; 11) $6\sqrt{2}$