Math 002 Multiple – Angle Identities

• • Double – Angle Identities : • $\sin 2x = 2\sin x\cos x$.

• $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$.
Remember that : $\sin^2 x + \cos^2 x = 1$, $-\sin^2 x - \cos^2 x = -1$ while $\sin^2 x - \cos^2 x = -\cos 2x$

• $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$. Notice that : $\dfrac{2\tan x}{1+\tan^2 x} = \dfrac{2\sin x/\cos x}{\sec^2 x} = 2\sin x\cos x = \sin 2x$.

In fact we can create many many identities from the above identities by just replacing x by any expression like $2x$, $3x$, $\frac{x}{2}$ ... etc. For example :

$$2\sin 3x\cos 3x = \sin 6x \quad , \quad 2\cos^2 4x - 1 = \cos 8x \quad , \quad \frac{2\tan\frac{x}{3}}{1-\tan^2\frac{x}{3}} = \tan\frac{2x}{3}.$$

Examples :

1) $\sin 22.5^\circ\cos 22.5^\circ = \frac{1}{2}\,(2\sin 22.5^\circ\cos 22.5^\circ) = \frac{1}{2}\,\sin 45^\circ = \frac{1}{2}\,\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}$.

2) $\dfrac{-6\tan 67.5^\circ}{1-\tan^2 67.5^\circ} = -3\cdot\dfrac{2\tan 67.5^\circ}{1-\tan^2 67.5^\circ} = -3\tan 2(67.5^\circ) = -3\tan 135^\circ = -3(-1) = 3$.

3) $\frac{1}{2} - \sin^2\frac{5\pi}{12} = \frac{1}{2}(1 - 2\sin^2\frac{5\pi}{12}) = \frac{1}{2}\cos\frac{5\pi}{6} = \frac{1}{2}(-\frac{\sqrt{3}}{2}) = -\frac{\sqrt{3}}{4}$.

4) $(\sin 3x + \cos 3x)^2 = \sin^2 3x + (2\sin 3x\cos 3x) + \cos^2 3x = (\sin^2 3x + \cos^2 3x) + \sin 6x = 1 + \sin 6x$.

5) $\sin^4 5x - \cos^4 5x = (\sin^2 5x + \cos^2 5x)\,(\sin^2 5x - \cos^2 5x) = (1)[-\cos 2(5x)] = -\cos 10x$.

• • Half – Angle Identities : • $\sin\frac{x}{2} = \pm\sqrt{\dfrac{1-\cos x}{2}}$ • $\cos\frac{x}{2} = \pm\sqrt{\dfrac{1+\cos x}{2}}$

• $\tan\frac{x}{2} = \pm\sqrt{\dfrac{1-\cos x}{1+\cos x}}$, $\tan\frac{x}{2} = \dfrac{\sin x}{1+\cos x}$ and also $\tan\frac{x}{2} = \csc x - \cot x$.

Notice that $\pm$ depends on the position of the angle $\frac{x}{2}$ Not x.

By squaring the above identities we can get : $\dfrac{1-\cos x}{2} = \sin^2\frac{x}{2}$, $\dfrac{1+\cos x}{2} = \cos^2\frac{x}{2}$, $\dfrac{1-\cos x}{1+\cos x} = \tan^2\frac{x}{2}$ and $\dfrac{2}{1-\cos x} = \csc^2\frac{x}{2}$, $\dfrac{2}{1+\cos x} = \sec^2\frac{x}{2}$, $\dfrac{1+\cos x}{1-\cos x} = \cot^2\frac{x}{2}$

Again we can play with these identities by replacing x by any expression. For example :

$$\sin x = \pm\sqrt{\frac{1-\cos 2x}{2}} \quad , \quad \cos\frac{x}{4} = \pm\sqrt{\frac{1+\cos\frac{x}{2}}{2}} \quad , \quad \tan\frac{x}{6} = \csc\frac{x}{3} - \cot\frac{x}{3}$$

Examples :

1) $\sin 22.5^\circ = \sin\dfrac{45^\circ}{2} = \sqrt{\dfrac{1-\cos 45^\circ}{2}} = \sqrt{\dfrac{1-\frac{\sqrt{2}}{2}}{2}} = \ldots = \dfrac{\sqrt{2-\sqrt{2}}}{2}$.

2) $\cos\frac{7\pi}{12} = \cos\frac{1}{2}\,\frac{7\pi}{6} = -\sqrt{\dfrac{1+\cos\frac{7\pi}{6}}{2}} = -\sqrt{\dfrac{1-\frac{\sqrt{3}}{2}}{2}} = \ldots = \dfrac{\sqrt{2-\sqrt{3}}}{2}$.

3) $\tan^2 157.5^\circ = \tan^2\dfrac{315^\circ}{2} = \dfrac{1-\cos 315^\circ}{1+\cos 315^\circ} = \dfrac{1-\sqrt{2}/2}{1+\sqrt{2}/2} = \ldots = 3 - 2\sqrt{2}$.

4) $\sqrt{\dfrac{1-\sin 460^\circ}{2}} = \sqrt{\dfrac{1-\sin(460^\circ - 360^\circ)}{2}} = \sqrt{\dfrac{1-\sin 100^\circ}{2}} = \sqrt{\dfrac{1-\cos(90^\circ - 100^\circ)}{2}} = \sqrt{\dfrac{1-\cos(-10^\circ)}{2}}$
$= |\sin\dfrac{-10^\circ}{2}| = \sin 5^\circ$

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