Math 002 —– Inverse Trig. Functions
Notes and Examples

1. Remember that $\mathrm{Sin}^{-1}x, \mathrm{Cos}^{-1}x$, . . . and $\mathrm{Sec}^{-1}x$ are all functions , this means that they have a single value for each x in their domain , e.g. , $\mathrm{Sin}^{-1}\frac{1}{2} = \frac{\pi}{6}$ **only** while the equation $\sin x = \frac{1}{2}$ has infinite number of solutions , namely $\{\frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi\}$, where n is any integer .

2. For $x \geq 0$ and it is within the domain of each function we have :
$\mathrm{Sin}^{-1}x$, $\mathrm{Cos}^{-1}x$, $\mathrm{Tan}^{-1}x$, $\mathrm{Sec}^{-1}x$, $\mathrm{Csc}^{-1}x$ and $\mathrm{Cot}^{-1}x$ are all $\in [0, \frac{\pi}{2}]$. e . g . , $\mathrm{Sin}^{-1}(1) = \frac{\pi}{2}$, $\mathrm{Cos}^{-1}(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$, $\mathrm{Cot}^{-1}(\sqrt{3}) = \frac{\pi}{6}$.

3. For $x \geq 0$ (so $-x \leq 0$) and it is within the domain of each function we have :

   (a) $\mathrm{Sin}^{-1}(-x) = -\mathrm{Sin}^{-1}x$, $\mathrm{Csc}^{-1}(-x) = -\mathrm{Csc}^{-1}x$ and $\mathrm{Tan}^{-1}(-x) = -\mathrm{Tan}^{-1}x$.
   This shows that $\mathrm{Sin}^{-1}x, \mathrm{Csc}^{-1}x$ and $\mathrm{Tan}^{-1}x$ are **odd** functions .

   (b) $\mathrm{Cos}^{-1}(-x) = \pi - \mathrm{Cos}^{-1}x$, $\mathrm{Sec}^{-1}(-x) = \pi - \mathrm{Sec}^{-1}x$ and $\mathrm{Cot}^{-1}(-x) = \pi - \mathrm{Cot}^{-1}x$.
   Clearly $\mathrm{Cos}^{-1}x, \mathrm{Sec}^{-1}x$ and $\mathrm{Cot}^{-1}x$ are **neither** odd **nor** even functions .

   The reason for this is the type of the **range** of each function .

   EX. : $\mathrm{Csc}^{-1}(-\sqrt{2}) = -\mathrm{Csc}^{-1}\sqrt{2} = -\frac{\pi}{4}$, $\mathrm{Sec}^{-1}(-2) = \pi - \mathrm{Sec}^{-1}2 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$, $\mathrm{Cot}^{-1}(-\sqrt{3}) = \pi - \mathrm{Cot}^{-1}\sqrt{3} = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

4. We can notice the following :

   (a) $\mathrm{Csc}^{-1}x = \mathrm{Sin}^{-1}\frac{1}{x}$ if $|x| \geq 1$

   (b) $\mathrm{Sec}^{-1}x = \mathrm{Cos}^{-1}\frac{1}{x}$ if $|x| \geq 1$

   (c) $\mathrm{Cot}^{-1}x = \mathrm{Tan}^{-1}\frac{1}{x}$ if $x > 0$

   (d) $\mathrm{Cot}^{-1}x = \pi + \mathrm{Tan}^{-1}\frac{1}{x}$ if $x < 0$

5. We have the following statements :

   (a) $\sin(\mathrm{Sin}^{-1}x) = x$ if $|x| \leq 1$

   (b) $\cos(\mathrm{Cos}^{-1}x) = x$ if $|x| \leq 1$

   (c) $\tan(\mathrm{Tan}^{-1}x) = x$ for any $x \in \Re$

   (d) $\csc(\mathrm{Csc}^{-1}x) = x$ if $|x| \geq 1$

   (e) $\sec(\mathrm{Sec}^{-1}x) = x$ if $|x| \geq 1$

   (f) $\cot(\mathrm{Cot}^{-1}x) = x$ for any $x \in \Re$

   EX. : $\sin(\mathrm{Sin}^{-1}\frac{1}{3}) = \frac{1}{3}$, $\tan(\mathrm{Tan}^{-1}(-5)) = -5$ while $\cos(\mathrm{Cos}^{-1}2)$, $\sec(\mathrm{Sec}^{-1}\frac{1}{2})$ and $\csc(\mathrm{Csc}^{-1}(\frac{-1}{3}))$ are **not defined** .

6. Similarly we have :

   (a) $\mathrm{Sin}^{-1}(\sin x) = x$ if $|x| \leq \frac{\pi}{2}$

   (b) $\mathrm{Cos}^{-1}(\cos x) = x$ if $0 \leq x \leq \pi$

   (c) $\mathrm{Tan}^{-1}(\tan x) = x$ if $|x| < \frac{\pi}{2}$

   (d) $\mathrm{Csc}^{-1}(\csc x) = x$ if $x \in [-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$

   (e) $\mathrm{Sec}^{-1}(\sec x) = x$ if $x \in [0, \pi] - \{\frac{\pi}{2}\}$

   (f) $\mathrm{Cot}^{-1}(\cot x) = x$ if $0 < x < \pi$

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EX. : $\text{Sin}^{-1}(\sin\frac{\pi}{5}) = \frac{\pi}{5}$ because $\frac{\pi}{5} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ while

$\text{Sin}^{-1}(\sin\frac{3\pi}{4}) = \text{Sin}^{-1}(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$ because $\frac{3\pi}{4} \notin [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\text{Cos}^{-1}(\cos\frac{5\pi}{6}) = \frac{5\pi}{6}$ because $\frac{5\pi}{6} \in [0, \pi]$ while

$\text{Cos}^{-1}(\cos\frac{4\pi}{3}) = \text{Cos}^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$ because $\frac{4\pi}{3} \notin [0, \pi]$

$\text{Tan}^{-1}(\tan(-\frac{\pi}{3})) = -\frac{\pi}{3}$ because $-\frac{\pi}{3} \in (-\frac{\pi}{2}, \frac{\pi}{2})$ while

$\text{Tan}^{-1}(\tan\frac{3\pi}{4}) = \text{Tan}^{-1}(-1) = -\frac{\pi}{4}$ because $\frac{3\pi}{4} \notin (-\frac{\pi}{2}, \frac{\pi}{2})$.

Problems :

1. If $f(x) = \cos(\text{Sin}^{-1}2x) + \text{Sec}^{-1}(\csc \pi x)$, find $f(\frac{1}{2}) + f(-\frac{1}{4})$.
   Ans : $f(\frac{1}{2}) = \cos(\text{Sin}^{-1}1) + \text{Sec}^{-1}(\csc\frac{\pi}{2}) = \cos\frac{\pi}{2} + \text{Sec}^{-1}1 = 0 + 0 = 0$ and
   $f(-\frac{1}{4}) = \cos(\text{Sin}^{-1}(-\frac{1}{2})) + \text{Sec}^{-1}(\csc(\frac{-\pi}{4})) = \cos(-\text{Sin}^{-1}\frac{1}{2}) + \text{Sec}^{-1}(-\sqrt{2})$
   $= \cos(-\frac{\pi}{6}) + (\pi - \text{Sec}^{-1}(\sqrt{2}) = \cos\frac{\pi}{6} + (\pi - \frac{\pi}{4}) = \frac{\sqrt{3}}{2} + \frac{3\pi}{4}$, then
   $f(\frac{1}{2}) + f(-\frac{1}{4}) = 0 + \frac{\sqrt{3}}{2} + \frac{3\pi}{4} = \frac{\sqrt{3}}{2} + \frac{3\pi}{4}$.

2. Find the value of : $\text{Sin}^{-1}(\sin\frac{89\pi}{5})$; $\text{Cos}^{-1}(\cos\frac{34\pi}{7})$; $\text{Tan}^{-1}(\tan\frac{105\pi}{8})$.
   Ans : First remember that the period of $\sin x$ and $\cos x$ is 2π while $\tan x$ has period π .
   $\text{Sin}^{-1}(\sin\frac{89\pi}{5}) = \text{Sin}^{-1}[\sin(18\pi - \frac{\pi}{5})] = \text{Sin}^{-1}[\sin(-\frac{\pi}{5})] = -\frac{\pi}{5} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.
   $\text{Cos}^{-1}(\cos\frac{34\pi}{7}) = \text{Cos}^{-1}[\cos(4\pi + \frac{6\pi}{7})] = \text{Cos}^{-1}[\cos(\frac{6\pi}{7})] = \frac{6\pi}{7} \in [0, \pi]$.
   $\text{Tan}^{-1}(\tan\frac{105\pi}{8}) = \text{Tan}^{-1}[\tan(13\pi + \frac{\pi}{8})] = \text{Tan}^{-1}[\tan(\frac{\pi}{8})] = \frac{\pi}{8} \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

3. Find $f^{-1}(x)$ if : $y = f(x) = 2\text{Sec}^{-1}(3x + \pi) + 5$.
   Ans : First we solve for x : $y - 5 = 2\text{Sec}^{-1}(3x + \pi) \Longrightarrow \text{Sec}^{-1}(3x + \pi) = \frac{y-5}{2}$
   $\Longrightarrow 3x + \pi = \sec(\frac{y-5}{2}) \Longrightarrow 3x = \sec(\frac{y-5}{2}) - \pi \Longrightarrow x = \frac{1}{3}[\sec(\frac{y-5}{2}) - \pi]$.
   Next we exchange x with y to get $y = f^{-1}(x) = \frac{1}{3}[\sec(\frac{x-5}{2}) - \pi]$.

4. Find the domain and range of the following :

   (a) $y = 2\text{Cos}^{-1}\frac{x}{3}$.
   Ans : Since the domain of arccosine function is $[-1, 1]$, then we have
   $-1 \leq \frac{x}{3} \leq 1 \Longrightarrow -3 \leq x \leq 3$. Thus the domain $= [-3, 3]$.
   We have $0 \leq \text{Cos}^{-1}\frac{x}{3} \leq \pi \Longrightarrow 0 \leq 2\text{Cos}^{-1}\frac{x}{3} \leq 2\pi$. Thus the range $= [0, 2\pi]$.

   (b) $y = -\text{Csc}^{-1}(x + 1) + \pi$.
   Ans : Since the domain of arccosecant function is $(-\infty, -1] \cup [1, \infty)$, then we have
   $x + 1 \leq -1$ or $x + 1 \geq 1 \Longrightarrow x \leq -2$ or $x \geq 0$.
   Thus the domain $= (-\infty, -2] \cup [0, \infty)$.
   We know that $-\frac{\pi}{2} \leq \text{Csc}^{-1}(x + 1) < 0$ or $0 < \text{Csc}^{-1}(x + 1) \leq \frac{\pi}{2}$
   $\Longrightarrow \frac{\pi}{2} \geq -\text{Csc}^{-1}(x + 1) > 0$ or $0 > -\text{Csc}^{-1}(x + 1) \geq -\frac{\pi}{2}$.
   By adding π to all sides we get the range $= [\frac{\pi}{2}, \pi) \cup (\pi, \frac{3\pi}{2}]$.

Comparison between Trig.Func.& their Inverses

1. **Domain & Range :**

Function	Domain	Range
$\sin x$	$\Re$	$[-1,1]$
$\cos x$	$\Re$	$[-1,1]$
$\tan x$	$\Re-\{(2n+1)\frac{\pi}{2}\}$	$\Re$
$\cot x$	$\Re-\{n\pi\}$	$\Re$
$\csc x$	$\Re-\{n\pi\}$	$(-\infty,-1]\cup[1,\infty)$
$\sec x$	$\Re-\{(2n+1)\frac{\pi}{2}\}$	$(-\infty,-1]\cup[1,\infty)$

Function	Domain	Range
$\mathrm{Sin}^{-1}x$	$[-1,1]$	$[-\frac{\pi}{2},\frac{\pi}{2}]$
$\mathrm{Cos}^{-1}x$	$[-1,1]$	$[0,\pi]$
$\mathrm{Tan}^{-1}x$	$\Re$	$(-\frac{\pi}{2},\frac{\pi}{2})$
$\mathrm{Cot}^{-1}x$	$\Re$	$(0,\pi)$
$\mathrm{Csc}^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$[-\frac{\pi}{2},\frac{\pi}{2}]-\{0\}$
$\mathrm{Sec}^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$[0,\pi]-\{\frac{\pi}{2}\}$

2. **Periodic Functions** :

   (a) $\sin x$, $\cos x$, $\sec x$, $\csc x$ are periodic functions of period 2π and $\tan x$, $\cot x$ are periodic functions of period π .

   (b) $\mathrm{Sin}^{-1}x$, $\mathrm{Cos}^{-1}x$,$\mathrm{Sec}^{-1}x$, $\mathrm{Csc}^{-1}x$, $\mathrm{Tan}^{-1}x$ and $\mathrm{Cot}^{-1}x$ are **not** periodic functions .

3. **$1-1$ Functions :**

   (a) $\sin x$, $\cos x$, $\sec x$, $\csc x$, $\tan x$ and $\cot x$ are **not** $1-1$ functions .

   (b) $\mathrm{Sin}^{-1}x$, $\mathrm{Cos}^{-1}x$, $\mathrm{Sec}^{-1}x$, $\mathrm{Csc}^{-1}x$, $\mathrm{Tan}^{-1}x$ and $\mathrm{Cot}^{-1}x$ are $1-1$ functions .

4. **Odd and Even Functions :**

   (a) $\sin x$, $\csc x$, $\tan x$ and $\cot x$ are odd functions , i . e . , $\sin(-x)=-\sin x$ etc , but $\cos x$, $\sec x$ are even functions , i . e . , $\cos(-x)=\cos x$ and $\sec(-x)=\sec x$.

   (b) $\mathrm{Sin}^{-1}x$, $\mathrm{Csc}^{-1}x$, $\mathrm{Tan}^{-1}x$ are odd functions but the others are neither odd nor even , e.g. , $\mathrm{Csc}^{-1}(-2)=-\mathrm{Csc}^{-1}(2)=-\frac{\pi}{6}$, $\mathrm{Cos}^{-1}(-\frac{1}{2})=\frac{2\pi}{3}\neq\mathrm{Cos}^{-1}(\frac{1}{2})$ and $\neq-\mathrm{Cos}^{-1}(\frac{1}{2})$.

5. **Identities :**

   (a) $\sec x=\frac{1}{\cos x}$, $\csc x=\frac{1}{\sin x}$, $\tan x=\frac{\sin x}{\cos x}$ and $\cot x=\frac{\cos x}{\sin x}$.

   (b) $\mathrm{Sec}^{-1}x\neq\frac{1}{\mathrm{Cos}^{-1}x}$, $\mathrm{Csc}^{-1}x\neq\frac{1}{\mathrm{Sin}^{-1}x}$, $\mathrm{Tan}^{-1}x\neq\frac{\mathrm{Sin}^{-1}x}{\mathrm{Cos}^{-1}x}$ and $\mathrm{Cot}^{-1}x\neq\frac{\mathrm{Cos}^{-1}x}{\mathrm{Sin}^{-1}x}$.

6. **Other Identities :** $\sin^2x+\cos^2x=1$ while $(\mathrm{Sin}^{-1}x)^2+(\mathrm{Cos}^{-1}x)^2\neq1$. Similarly for the other Identities .

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Table

Let us write $\bullet$ for Undefined and $\star$ for by Calculator .

$\mathbf{x}$	$\mathrm{Sin}^{-1}\mathbf{x}$	$\mathrm{Cos}^{-1}\mathbf{x}$	$\mathrm{Tan}^{-1}\mathbf{x}$	$\mathrm{Csc}^{-1}\mathbf{x}$	$\mathrm{Sec}^{-1}\mathbf{x}$	$\mathrm{Cot}^{-1}\mathbf{x}$
0	0	$\frac{\pi}{2}$	0	$\bullet$	$\bullet$	$\frac{\pi}{2}$
$\frac{1}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\star$	$\bullet$	$\bullet$	$\star$
$\frac{1}{\sqrt{3}}$	$\star$	$\star$	$\frac{\pi}{6}$	$\bullet$	$\bullet$	$\frac{\pi}{3}$
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\star$	$\bullet$	$\bullet$	$\star$
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{6}$	$\star$	$\bullet$	$\bullet$	$\star$
1	$\frac{\pi}{2}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	0	$\frac{\pi}{4}$
$\frac{2}{\sqrt{3}}$	$\bullet$	$\bullet$	$\star$	$\frac{\pi}{3}$	$\frac{\pi}{6}$	$\star$
$\sqrt{2}$	$\bullet$	$\bullet$	$\star$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\star$
$\sqrt{3}$	$\bullet$	$\bullet$	$\frac{\pi}{3}$	$\star$	$\star$	$\frac{\pi}{6}$
2	$\bullet$	$\bullet$	$\star$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\star$
$-\frac{1}{2}$	$-\frac{\pi}{6}$	$\frac{2\pi}{3}$	$\star$	$\bullet$	$\bullet$	$\star$
$-\frac{1}{\sqrt{3}}$	$\star$	$\star$	$-\frac{\pi}{6}$	$\bullet$	$\bullet$	$\frac{2\pi}{3}$
$-\frac{\sqrt{2}}{2}$	$-\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\star$	$\bullet$	$\bullet$	$\star$
$-\frac{\sqrt{3}}{2}$	$-\frac{\pi}{3}$	$\frac{5\pi}{6}$	$\star$	$\bullet$	$\bullet$	$\star$
-1	$-\frac{\pi}{2}$	π	$-\frac{\pi}{4}$	$-\frac{\pi}{2}$	π	$\frac{3\pi}{4}$
$-\frac{2}{\sqrt{3}}$	$\bullet$	$\bullet$	$\star$	$-\frac{\pi}{3}$	$\frac{5\pi}{6}$	$\star$
$-\sqrt{2}$	$\bullet$	$\bullet$	$\star$	$-\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\star$
$-\sqrt{3}$	$\bullet$	$\bullet$	$-\frac{\pi}{3}$	$\star$	$\star$	$\frac{5\pi}{6}$
-2	$\bullet$	$\bullet$	$\star$	$-\frac{\pi}{6}$	$\frac{2\pi}{3}$	$\star$

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