# MATH 002 – DOMAIN & RANGE

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| Let $a > 0$ , $a \neq 1$ , n any integer and $\Re$ be the set of real numbers | | | | |
|---|---|---|---|---|
| **Function** | **Domain** | **Range** | **Asymptote(s)** | **Comments** |
| $y = a^x$ | $\Re$ | $(0, \infty)$ | $y = 0$ | |
| $y = -a^x$ | $\Re$ | $(-\infty, 0)$ | $y = 0$ | $-a^x \neq (-a)^x$ |
| $y = a^{x^2}$ , $a > 1$ | $\Re$ | $[1, \infty)$ | None | See its graph |
| $y = a^{-x^2}$ , $a > 1$ | $\Re$ | $(0, 1]$ | $y = 0$ | $y = a^{-x^2}, a > 1$ same as $y = a^{x^2}, 0 < a < 1$ |
| $y = a^{\|x\|}$ , $a > 1$ | $\Re$ | $[1, \infty)$ | None | See its graph |
| $y = a^{-\|x\|}$ , $a > 1$ | $\Re$ | $(0, 1]$ | $y = 0$ | See its graph |
| $y = \log_a x$ | $(0, \infty)$ | $\Re$ | $x = 0$ | |
| $y = -\log_a x$ | $(0, \infty)$ | $\Re$ | $x = 0$ | $y = \pm \log x$ are symm. w.r.t. x–axis |
| $y = \log_a(-x)$ | $(-\infty, 0)$ | $\Re$ | $x = 0$ | |
| $y = 2\log_a x$ | $(0, \infty)$ | $\Re$ | $x = 0$ | $2\log x = \log x^2$ if $x > 0$ |
| $y = \log_a x^2$ | $(-\infty, 0) \cup (0, \infty)$ | $\Re$ | $x = 0$ | $\log x^2 = 2\log \|x\|$ |
| $y = \log_a \|x\|$ | $(-\infty, 0) \cup (0, \infty)$ | $\Re$ | $x = 0$ | |
| $y = \| \log_a x \|$ | $(0, \infty)$ | $[0, \infty)$ | $x = 0$ | |
| $y = \left\| \log_a \|x\| \right\|$ | $(-\infty, 0) \cup (0, \infty)$ | $[0, \infty)$ | $x = 0$ | |
| $y = \sin x$ | $\Re$ | $[-1, 1]$ | None | |
| $y = \cos x$ | $\Re$ | $[-1, 1]$ | None | |
| $y = \csc x$ | $\Re - \{n\pi\}$ | $(-\infty, -1] \cup [1, \infty)$ | $x = n\pi$ | |
| $y = \sec x$ | $\Re - \{(2n+1)\frac{\pi}{2}\}$ | $(-\infty, -1] \cup [1, \infty)$ | $x = (2n+1)\frac{\pi}{2}$ | |
| $y = \tan x$ | $\Re - \{(2n+1)\frac{\pi}{2}\}$ | $\Re$ | $x = (2n+1)\frac{\pi}{2}$ | |
| $y = \cot x$ | $\Re - \{n\pi\}$ | $\Re$ | $x = n\pi$ | |
| $y = \text{Sin}^{-1}x$ | $[-1, 1]$ | $[-\frac{\pi}{2}, \frac{\pi}{2}]$ | None | |
| $y = \text{Cos}^{-1}x$ | $[-1, 1]$ | $[0, \pi]$ | None | |
| $y = \text{Tan}^{-1}x$ | $\Re$ | $(-\frac{\pi}{2}, \frac{\pi}{2})$ | $y = \pm\frac{\pi}{2}$ | |
| $y = \text{Csc}^{-1}x$ | $(-\infty, -1] \cup [1, \infty)$ | $[-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$ | $y = 0$ | |
| $y = \text{Sec}^{-1}x$ | $(-\infty, -1] \cup [1, \infty)$ | $[0, \pi] - \{\frac{\pi}{2}\}$ | $y = \frac{\pi}{2}$ | |
| $y = \text{Cot}^{-1}x$ | $\Re$ | $(0, \pi)$ | $y = 0\,;\, y = \pi$ | |

**Note** : To find the domain and range of more complicated functions involving exponential , logarithmic , trigonometric or inverse trigonometric functions such as $y = \log(x^2 - 3) + 1$ ; $y = 2\text{Sin}^{-1}(x+2) - \pi$ :

- First : You should remember the domain and range of all the functions listed in the table .
- For the domain : Solve for $x$ if $q(x) \in \text{D}$ , where $q(x)$ is the expression containing $x$ and just coming after the function , D is the domain of the original function . For example when we have $y = \log(x+3) - 1$ , then $q(x) = x + 3 \in (0, \infty)$ , i.e. , $x + 3 > 0 \implies x > -3$ . For $y = 2\text{Cos}^{-1}(2x-1) + \pi$ , then $q(x) = 2x - 1 \in [-1, 1]$ , i.e., $-1 \le 2x - 1 \le 1 \implies 0 \le x \le 1$ .
- For the range : Keep in mind the range of the original function and try to build the new shaped function from there . To make it more clear suppose the range of the original function $y_1$ is $[a, b]$, then to find the range of $y = f(x) = 2y_1 - 3$ , start from $a \le y_1 \le b$ , then multiply by 2 to get $2a \le 2y_1 \le 2b$ , finally add $-3$ to all sides to have $2a - 3 \le 2y_1 - 3 \le 2b - 3$ . So the range of $y = f(x)$ is $[2a - 3, 2b - 3]$ . Next let us find the domain and range of the following functions.

**Example 1** : $y = 3e^{x+1} - 4$ . Here the domain $= \Re$ while the range $= (-4, \infty)$ because $3e^{x+1} > 0 \implies 3e^{x+1} - 4 > -4$ .

**Example 2** : $y = -(\frac{1}{2})^{(3-2x)} + 3$ . Again the domain $= \Re$ , the range $= (-\infty, 3)$ because $-(\frac{1}{2})^{(3-2x)} < 0 \implies -(\frac{1}{2})^{(3-2x)} + 3 < 3$ .

**Example 3** : $y = 5^{-x^2} - 4$ . The domain $= \Re$ and from the table we have $0 < 5^{-x^2} \le 1 \implies -4 < 5^{-x^2} - 4 \le -3$ , so the range $= (-4, -3]$ .

**Example 4** : $y = -3^{x^2} + 6$ . The domain $= \Re$ and again from the table we have $3^{x^2} \ge 1 \implies -3^{x^2} \le -1 \implies -3^{x^2} + 6 \le 5$ , so the range $= (-\infty, 5]$ .

**Example 5** : $y = 2\log(1 - 2x) - 5$ . Remember the above note which says $1 - 2x > 0 \implies -2x > -1 \implies x < \frac{1}{2}$ , then the domain $= (-\infty, \frac{1}{2})$ . The range $= \Re$ .

**Example 6** : $y = \log|x + 3| - 3$ . Since $|x + 3| > 0$ for all real numbrs except $x = -3$ , then the domain $= (-\infty, -3) \cup (-3, \infty)$ while the range $= \Re$ .

**Example 7** : $y = -\ln(x^2 - 4) + 1$ . As before $(x^2 - 4) > 0 \implies x^2 > 4 \implies |x| > 2 \implies x < -2$ or $x > 2$ , then the domain $= (-\infty, -2) \cup (2, \infty)$ , the range $= \Re$ .

**Example 8** : $y = \Big| \log|x - 2| \Big| - 3$ . Here $|x - 2| > 0$ for all real numbers except $x = 2$ , then the domain $= (-\infty, 2) \cup (2, \infty)$ . Now $\Big| \log|x - 2| \Big| \ge 0 \implies \Big| \log|x - 2| \Big| - 3 \ge -3$ , then the range $= [-3, \infty)$ .

**Example 9** : $y = -\Big| \ln(-4 - 2x) \Big| + 4$ . We have $-4 - 2x > 0 \implies -2x > 4 \implies x < -2 \implies$ the domain $= (-\infty, -2)$ . Since $-\Big| \ln(-4 - 2x) \Big| \le 0 \implies -\Big| \ln(-4 - 2x) \Big| + 4 \le 4$ , so the range $= (-\infty, 4]$ .

**Example 10** : $y = 3\sin(4x-\pi)+2$ . The domain $= \Re$ and the range $= [-3+2, 3+2] = [-1, 5]$, because $-1 \le \sin(4x-\pi) \le 1 \implies -3 \le 3\sin(4x-\pi) \le 3 \implies -1 \le 3\sin(4x-\pi)+2 \le 5$ .

**Example 11** : $y = -2\sec(3x+\pi)+5$ . Recall that the domain of $\sec x$ is $\Re - \{(2n+1)\frac{\pi}{2}\}$ , then for our new function we have $3x+\pi \ne (2n+1)\frac{\pi}{2} \implies 3x \ne (2n+1)\frac{\pi}{2} - \pi \implies$ $x \ne (2n+1)\frac{\pi}{6} - \frac{\pi}{3} = [2n+1-2]\frac{\pi}{6} = (2n-1)\frac{\pi}{6} \implies$ the domain $= \Re - \{(2n-1)\frac{\pi}{6}\}$ . Remember that $\sec(3x+\pi) \le -1$ or $\sec(3x+\pi) \ge 1 \implies -2\sec(3x+\pi) \ge 2$ or $-2\sec(3x+\pi) \le -2$ $\implies -2\sec(3x+\pi)+5 \ge 7$ or $-2\sec(3x+\pi)+5 \le 3 \implies$ the range $= (-\infty, 3] \cup [7, \infty)$ .

**Example 12** : $y = 3\cot(2x - \frac{\pi}{4})+1$ . Recall that the domain of $\cot x$ is $\Re - \{n\pi\}$ , therefore $2x - \frac{\pi}{4} \ne n\pi \implies 2x \ne n\pi + \frac{\pi}{4} \implies x \ne n\frac{\pi}{2} + \frac{\pi}{8} = (4n+1)\frac{\pi}{8} \implies$ the domain $= \Re - \{(4n+1)\frac{\pi}{8}\}$. Since the range of $\cot x$ is $\Re$ , then the the range of this function is still $\Re$ .

**Example 13** : $y = 2|\cos(3x+\pi)| - 3$ . The domain $= \Re$ . Since $-2 \le 2\cos(3x+\pi) \le 2$ , then $0 \le 2|\cos(3x+\pi)| \le 2 \implies -3 \le 2|\cos(3x+\pi)| - 3 \le -1 \implies$ the range $= [-3, -1]$ .

**Example 14** : $y = 4\ \mathrm{Sin}^{-1}(5-3x)+\pi$ . Recall the domain of $\mathrm{Sin}^{-1}x$ is $[-1, 1]$ , then $-1 \le 5-3x \le 1 \implies -6 \le -3x \le -4 \implies 2 \ge x \ge \frac{4}{3} \implies$ the domain $= [\frac{4}{3}, 2]$ . Now $-\frac{\pi}{2} \le \mathrm{Sin}^{-1}(5-3x) \le \frac{\pi}{2} \implies -2\pi \le 4\ \mathrm{Sin}^{-1}(5-3x) \le 2\pi \implies -\pi \le 4\ \mathrm{Sin}^{-1}(5-3x)+\pi \le 3\pi$ $\implies$ the range $= [-\pi, 3\pi]$ .

**Example 15** : $y = \frac{3}{\pi}\ \mathrm{Cot}^{-1}(x+2)+1$ . The domain of $y = \mathrm{Cot}^{-1}x$ and this function is $\Re$ . Next we have $0 < \mathrm{Cot}^{-1}(x+2) < \pi \implies 0 < \frac{3}{\pi}\ \mathrm{Cot}^{-1}(x+2) < 3 \implies 1 < \frac{3}{\pi}\ \mathrm{Cot}^{-1}(x+2)+1 < 4$ $\implies$ the range $= (1, 4)$ .

**Example 16** : $y = -\mathrm{Sec}^{-1}(x^2-1)$ . Recall the domain of $\mathrm{Sec}^{-1}x$ is $(-\infty, -1] \cup [1, \infty)$ , then $x^2-1 \le -1$ or $x^2-1 \ge 1 \implies x^2 \le 0$ or $x^2 \ge 2 \implies x = 0$ or $|x| \ge \sqrt{2} \implies x = 0$ or $x \le -\sqrt{2}$ or $x \ge \sqrt{2} \implies$ the domain $= (-\infty, -\sqrt{2}] \cup \{0\} \cup [\sqrt{2}, \infty)$ . For the range we have $0 \le \mathrm{Sec}^{-1}(x^2-1) < \frac{\pi}{2}$ or $\frac{\pi}{2} < \mathrm{Sec}^{-1}(x^2-1) \le \pi \implies 0 \ge -\mathrm{Sec}^{-1}(x^2-1) > -\frac{\pi}{2}$ or $-\frac{\pi}{2} > -\mathrm{Sec}^{-1}(x^2-1) \ge -\pi \implies$ the range $= [-\pi, -\frac{\pi}{2}) \cup (-\frac{\pi}{2}, 0]$ .

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