Math 001 – Solved Problems On Chapter 1
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1. Find the solution set of the equations : (i) $\dfrac{x-3}{x+1} = \dfrac{2}{x+1} - \dfrac{6}{x+1}$ (ii) $\dfrac{x^2-6}{x-2} = \dfrac{2}{2-x}$

Ans: (i) Notice that $x \neq -1$. The equation is $\dfrac{x-3}{x+1} = \dfrac{2-6}{x+1} = \dfrac{-4}{x+1}$

multiply both sides by $(x+1)$ to get $x-3=-4 \implies x=-1$ but $x \neq -1 \implies$ S.S. $= \phi$

(ii) Again notice that $x \neq 2$. The equation is $\dfrac{x^2-6}{x-2} = \dfrac{-2}{x-2} \implies x^2 - 6 = -2$

$\implies x^2 = 4 \implies x = \pm 2$ but $x \neq 2 \implies$ S.S. $= \{-2\}$

2. Solve the equation: $|x-3| + 4|2x-6| = 0$

Ans: $|x-3| + 4(2)|x-3| = 0 \implies |x-3| + 8|x-3| = 0 \implies 9|x-3| = 0$

$\implies |x-3| = 0 \implies x-3=0 \implies x=3$

3. Solve the following equations: (i) $|3x-6| + 6 - 3x = 0$ (ii) $|x-5| + x - 5 = 0$

Ans: (i) $3|x-2| = 3x-6 = 3(x-2) \implies |x-2| = x-2$ is always true when $x-2 \geq 0$

$\implies x \geq 2 \implies$ S.S. $= [2, \infty)$

(ii) $|x-5| = 5-x = -(x-5)$ is always true when $x-5 \leq 0 \implies x \leq 5$

$\implies$ S.S. $= (-\infty, 5]$

4. Solve the equation: $|x-2| + |-x-1| + 3 = 0$

Ans: $|x-2| + |-x-1| = -3$ is impossible, because the left side is non-negative while the right side is negative. Therefore S.S. $= \phi$

5. Find the solution set of the equation $2|2x^2-5| - 6 = 0$

Ans: $2|2x^2-5| = 6 \implies |2x^2-5| = 3 \implies 2x^2-5=3$ or $2x^2-5=-3$

$\implies 2x^2 = 8$ or $2x^2 = 2 \implies x^2 = 4$ or $x^2 = 1 \implies x = \pm 2$ or $x = \pm 1$

$\implies$ S.S. $= \{-2, 2, -1, 1\}$

6. Find the sum of all solutions of the equation $\left|\frac{3x+1}{x-3}\right| = 2$

Ans: $\frac{3x+1}{x-3} = 2$ or $\frac{3x+1}{x-3} = -2$, $x \neq 3 \implies 3x+1 = 2x-6$ or $3x+1 = -2x+6$
$\implies x = -7$ or $5x = 5 \implies x = -7$ or $x = 1$.
The sum of all solutions $= (-7) + 1 = -6$

7. Find the product of all solutions of the equation $\frac{|x-1|+2}{1+|x-1|} - \frac{3}{2} = 0$

Ans: $\frac{|x-1|+2}{1+|x-1|} = \frac{3}{2} \implies 2|x-1| + 4 = 3 + 3|x-1| \implies 1 = |x-1| \implies x-1 = 1$
or $x-1 = -1 \implies x = 2$ or $x = 0$. The product of all solutions $= (2)(0) = 0$

8. Solve the equation: $|2x-1| - 3x - 1 = 0$

Ans: In this type of absolute value equations, we must check our answer. $|2x-1| = 3x+1$
$\implies 2x-1 = 3x+1$ or $2x-1 = -(3x+1) = -3x-1 \implies x = -2$ or $5x = 0 \implies x = 0$.
Now, we check, for $x = -2$: $|-5| + 6 - 1 \stackrel{?}{=} 0$ (F=false), so -2 is rejected.
For $x = 0$: $|-1| - 0 - 1 \stackrel{?}{=} 0$ (T=true). Thus S.S. $= \{0\}$

9. Which one of the following equations is a **contradiction**?

(a) $3x - 4 = 2(x-2)$ (b) $|2-4x| = 2|2x-1|$ ✓(c) $(2x-3)^2 = (2x-3)(2x+3) - 12x$
(d) $x^3 - x = 0$ (e) $3x - 7 = 3(x-2) - 1$

Ans: (a) $3x - 4 = 2x - 4 \implies x = 0$, it is a conditional equation. (b) $2|1-2x| = 2|2x-1| \implies |1-2x| = |2x-1|$. This is an equation which is always true, i.e. , it is an identity.
(c) $4x^2 - 12x + 9 = 4x^2 - 9 - 12x \implies 9 = -9$ which is impossible, so it is a contradiction.
(d) $x(x^2 - 1) = 0 \implies x(x-1)(x+1) = 0 \implies x = 0, x = 1$ or $x = -1$. Again this one is a conditional equation. (e) $3x-7 = 3x-6-1 = 3x-7$, the two sides are identical, it is an identity.

10. If the two equations $2(x-1) + 5 = x$ and $kx + 4 = 2x$ are **equivalent**, find k.

Ans: The two equations have the same solution. The first one $2x-2+5 = x \implies x = -3$ which will satisfy the second equation, so $k(-3) + 4 = 2(-3) \implies -3k = -10 \implies k = \frac{-10}{-3} = \frac{10}{3}$.

11. Solve for x if $2x^{-1} + 3y^{-1} - 4z^{-1} = 0$

Ans: $\frac{2}{x} + \frac{3}{y} - \frac{4}{z} = 0 \implies \frac{2}{x} = \frac{4}{z} - \frac{3}{y} = \frac{4y-3z}{yz} \implies \frac{x}{2} = \frac{yz}{4y-3z} \implies x = \frac{2yz}{4y-3z}$

12. Solve for x if $\dfrac{x+y}{m} = \dfrac{3-2x}{k}$

Ans: Try to seperate x on one side. So $k(x+y) = m(3-2x) \implies kx + ky = 3m - 2mx$ $\implies kx + 2mx = 3m - ky \implies (k+2m)x = 3m - ky \implies x = \dfrac{3m-ky}{k+2m}$

13. One-half of a number plus one-third of the number is 35 less than twice the number. Find this number.

Ans: Let the number be x , so twice the number will be $2x$. Now $\frac{1}{2}x + \frac{1}{3}x = 2x - 35$ (multiply by 6 to get) $3x + 2x = 12x - 210 \implies 210 = 12x - 5x$ $= 7x \implies x = \dfrac{210}{7} = 30$. The number is 30.

14. The length of a rectangle is 3 ft less than twice the width of the rectangle. If the perimeter of the rectangle is 174 ft, find its area.

Ans: Let L be the length and W be the width of the rectangle. Given $\boxed{\mathrm{L} = 2\mathrm{W} - 3}$, the perimeter $= 174 = 2\mathrm{L} + 2\mathrm{W} \implies 87 = \mathrm{L} + \mathrm{W} = (2\mathrm{W} - 3) + \mathrm{W} = 3\mathrm{W} - 3$ $\implies 3\mathrm{W} = 90 \implies \mathrm{W} = 30$ ft and $\mathrm{L} = 2(30) - 3 = 57$ ft. Therefore the area $= \mathrm{LW} = (57)(30) = 1710\ \mathrm{ft}^2$

15. The perimeter of a triangle is 60 cm. Each of the two small sides is $\frac{1}{3}$ the length of the longest side. Find the length of each side.

Ans: Let L be the length of the longest side, so the the two smaller sides will be of length $\dfrac{\mathrm{L}}{3}$.
The perimeter $= 60 = \mathrm{L} + \dfrac{\mathrm{L}}{3} + \dfrac{\mathrm{L}}{3} = \dfrac{5\mathrm{L}}{3} \implies \mathrm{L} = \dfrac{(3)(60)}{5} = 36$ cm.
Thus the three sides are of length: 36 cm , 12 cm and 12 cm.

16. The denominator of a fraction is 5 more than half the numerator. If the numerator is increased by 20 and the denominator is decreased by 4, the resulting numer is 8. What is the original fraction.

Ans: Assume the original fraction $= \dfrac{n}{d}$, where $d = \dfrac{n}{2} + 5 \implies$ the original fraction $= \dfrac{n}{\frac{n}{2}+5}$.

After the changes, the new fraction $= \dfrac{n+20}{\left(\frac{n}{2}+5\right) - 4} = \dfrac{n+20}{\frac{n}{2}+1} = 8 \implies \dfrac{2n+40}{n+2} = 8 \implies$

$2n + 40 = 8n + 16 \implies 24 = 6n \implies n = 4 \implies$ the original fraction $= \dfrac{4}{2+5} = \dfrac{4}{7}$.

17. If the diameter of a big circle is three times the diameter of a small circle, then find the ratio of the area of the small circle to the area of the big circle.

Ans: Let r_1 , D_1 , A_1 be the radius , diameter , and area of the small circle and let r_2 , D_2 , A_2 be the radius , diameter , and area of the big circle. Given $D_2 = 3D_1 \implies$

$2r_2 = 3(2r_1) \implies \boxed{r_2 = 3r_1}$. The required ratio $= \dfrac{A_1}{A_2} = \dfrac{\pi r_1^2}{\pi r_2^2} = \dfrac{r_1^2}{r_2^2} = \dfrac{r_1^2}{(3r_1)^2} = \dfrac{r_1^2}{9r_1^2} = \dfrac{1}{9}$

18. If $\dfrac{1}{3}$ is one solution of the quadratic equation $6x^2 + mx - 1 = 0$, then find the **other** solution.

Ans: $x = \dfrac{1}{3}$ satisfies the equation, so $6\left(\dfrac{1}{9}\right) + m\left(\dfrac{1}{3}\right) - 1 = 0 \implies \dfrac{2}{3} + \dfrac{m}{3} = 1 \implies \dfrac{m}{3} = 1 - \dfrac{2}{3}$ $= \dfrac{1}{3} \implies m = 1$. The quadratic equation is $6x^2 + x - 1 = 0 \implies (3x-1)(2x+1) = 0$ $\implies x = \dfrac{1}{3}$ or $x = -\dfrac{1}{2}$. Therefore the the other solution is $-\dfrac{1}{2}$

19. Find the **product** of all solutions of the quadratic equation $ax^2 - 2x + 10 = 0$ whose **discriminant** is -36.

Ans: Discriminant $= -36 = b^2 - 4ac = 4 - 4(a)(10) \implies 40a = 40 \implies a = 1$. The quadratic equation is $x^2 - 2x + 10 = 0 \implies x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{2 \pm \sqrt{-36}}{2} = \dfrac{2 \pm 6i}{2} = 1 \pm 3i$.

Thus the product of all solutions $= (1+3i)(1-3i) = 1 + 9 = 10$

(or we can say the product of all solutions $= \dfrac{c}{a} = \dfrac{10}{1} = 10$)

20. Find the value(s) of k if the quadratic equation $x^2 + 2kx + 2x + 3 = 0$ has two equal real roots.

Ans: "two equal real roots" means "one repeated real root". This means that the equation $x^2 + 2(k+1)x + 3 = 0$ has discriminant $= 0 \implies b^2 - 4ac = 4(k+1)^2 - 4(1)(3) = 0$ $\implies 4(k+1)^2 = 12 \implies (k+1)^2 = 3 \implies k + 1 = \pm\sqrt{3} \implies k = -1 \pm \sqrt{3}$

21. Find the value(s) of m if the quadratic equation $2x^2 - 3x + 2m = 1$ has two equal real roots.

Ans: The **discriminant** of the equation $2x^2 - 3x + (2m-1) = 0$ is 0

$\implies b^2 - 4ac = 9 - 4(2)(2m-1) = 0 \implies 9 - 16m + 8 = 0 \implies 17 = 16m \implies m = \dfrac{17}{16}$.

22. Find the quadratic equation whose roots are $2 + i\sqrt{3}$, $2 - i\sqrt{3}$.

Ans: The quadratic equation is $x^2 - (r_1 + r_2)x + r_1 r_2 = 0$, where $r_1 + r_2 =$ sum of roots $= (2 + i\sqrt{3}) + (2 - i\sqrt{3}) = 4$ and $r_1 \cdot r_2 =$ product of roots $= (2 + i\sqrt{3})(2 - i\sqrt{3}) = 4 + 3 = 7$.

The quadratic equation is $x^2 - 4x + 7 = 0$

23. Find the quadratic equation whose roots are **double** the roots of the equation $2x^2 - 5x - 8 = 0$.

Ans: If r_1 , r_2 are the roots of the equation $2x^2 - 5x - 8 = 0$, then $r_1 + r_2 = -\frac{b}{a} = \frac{5}{2}$ and $r_1 \cdot r_2 = \frac{c}{a} = \frac{-8}{2} = -4$. The quadratic equation with double roots $2r_1$ and $2r_2$ is $x^2 - (2r_1 + 2r_2)x + (2r_1)(2r_2) = 0 \implies x^2 - 2(r_1 + r_2)x + 4r_1r_2 = 0 \implies x^2 - 2\left(\frac{5}{2}\right)x + 4(-4) = 0 \implies x^2 - 5x - 16 = 0$

24. If r_1 , r_2 are the roots of the quadratic equation $2x^2 - 3x + 4 = 0$. Find the quadratic equation whose roots are $\frac{1}{r_1}$, $\frac{1}{r_2}$.

Ans: For $2x^2 - 3x + 4 = 0$, $r_1 + r_2 = -\frac{b}{a} = \frac{3}{2}$ and $r_1 \cdot r_2 = \frac{c}{a} = \frac{4}{2} = 2$.

The quadratic equation with roots $\frac{1}{r_1}$, $\frac{1}{r_2}$ is $x^2 - \left(\frac{1}{r_1} + \frac{1}{r_2}\right)x + \frac{1}{r_1} \cdot \frac{1}{r_2} = 0 \implies$

$x^2 - \left(\frac{r_2 + r_1}{r_1 r_2}\right)x + \frac{1}{r_1 r_2} = 0 = x^2 - \frac{\frac{3}{2}}{2}x + \frac{1}{2} = 0 \implies x^2 - \frac{3}{4}x + \frac{1}{2} = 0 \implies 4x^2 - 3x + 2 = 0$

25. If the sides of a right triangle are of length $x + 8$, $x + 7$ and x cm , then find the area of the triangle.

Ans: $x + 8$ is the length of the longest side (hypotenuse). By

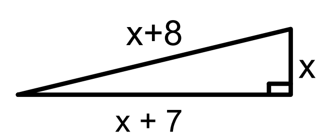
Pythagorean Theorem $(x+8)^2 = (x+7)^2 + x^2 \implies x^2 + 16x + 64 =$

$x^2 + 14x + 49 + x^2 \implies x^2 - 2x - 15 = 0 \implies (x - 5)(x + 3) = 0$

$\implies x = 5$ or $x = -3$ (rejected, because $x > 0$).

Therefore the sides are of length 13 , 12 and 5 cm.

The area $= \frac{1}{2}$ (base)(height) $= \frac{1}{2}(12)(5) = 30$ cm^2



26. If M , N , and L are three **consecutive positive even integers** such that the sum of their squares is 116, then M + N + L =

(a) 22 ✔(b) 18 (c) 40 (d) 16 (e) 32

Ans: Let the numbers be x , $x + 2$, $x + 4$. So $x^2 + (x + 2)^2 + (x + 4)^2 = 116 \implies$

$x^2 + x^2 + 4x + 4 + x^2 + 8x + 16 = 116 \implies 3x^2 + 12x - 96 = 0 \implies x^2 + 4x - 32 = 0$

$\implies (x - 4)(x + 8) = 0 \implies x = 4$ or $x = -8$ (rejected). Thus the numbers are

$\text{M} = x = 4$, $\text{N} = x + 2 = 6$, $\text{L} = x + 4 = 8 \implies \text{M} + \text{N} + \text{L} = 4 + 6 + 8 = 18$

27. If the equation $3x^2 - 2x + 1 = 0$ is written in the form $(x-a)^2 = b$, then $a + b =$

(a) $\frac{1}{3}$ (b) $-\frac{2}{9}$ (c) $-\frac{2}{3}$ (d) $\frac{3}{2}$ ✓(e) $\frac{1}{9}$

Ans: It can be done by completing the square. Divide by 3 to get $x^2 - \frac{2}{3}x = -\frac{1}{3}$, then add

$\left[\frac{1}{2}(-\frac{2}{3})\right]^2 = \frac{1}{9}$ to both sides $x^2 - \frac{2}{3}x + \frac{1}{9} = -\frac{1}{3} + \frac{1}{9} = -\frac{2}{9} \implies \left(x - \frac{1}{3}\right)^2 = -\frac{2}{9}$

$\implies a = \frac{1}{3}$, $b = -\frac{2}{9} \implies a + b = \frac{1}{3} - \frac{2}{9} = \frac{1}{9}$

28. Find all solutions of the equation $\left|2x^2 - 3x\right| - 9 = 0$

Ans: $\left|2x^2 - 3x\right| = 9 \implies 2x^2 - 3x = 9$ or $2x^2 - 3x = -9$

$\implies 2x^2 - 3x - 9 = 0 \implies (2x+3)(x-3) = 0 \implies x = -\frac{3}{2}$ or $x = 3$

or $2x^2 - 3x + 9 = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3 \pm \sqrt{9 - 72}}{4} = \frac{3 \pm \sqrt{-63}}{4} = \frac{3 \pm 3\sqrt{7}i}{4}$

$\implies$ S.S. $= \left\{-\frac{3}{2}, 3, \frac{3 + 3\sqrt{7}i}{4}, \frac{3 - 3\sqrt{7}i}{4}\right\}$

29. Solve the equation $|x|^2 - 2|x| = 3$

Ans: Let $y = |x|$, so the new equation is $y^2 - 2y - 3 = 0 \implies (y-3)(y+1) = 0$

$\implies y = 3$ or $y = -1 \implies |x| = 3 \implies x = \pm 3$ or $|x| = -1$ is impossible, because $|x| \geq 0$.

Therefore S.S. $= \{-3, 3\}$

30. Solve the equation $|x-3|^2 + 4|x-3| + 4 = 0$

Ans: Let $u = |x-3|$, the new equation is $u^2 + 4u + 4 = 0 \implies (u+2)^2 = 0$

$\implies u + 2 = 0 \implies u = -2 \implies |x-3| = -2$ which is impossible. Thus S.S. $= \phi$

31. Find all **real** solutions of the equation $|x^2 - 1|^2 - 5|x^2 - 1| + 6 = 0$

Ans: Our substitution is $y = |x^2 - 1|$ which leads to the equation $y^2 - 5y + 6 = 0$.

Don't forget that we are searching for the real roots only. By factoring, $(y-2)(y-3) = 0$

$\implies y = 2$ or $y = 3 \implies |x^2 - 1| = 2$ or $|x^2 - 1| = 3$. Now

$|x^2 - 1| = 2 \implies x^2 - 1 = 2$ or $x^2 - 1 = -2 \implies x^2 = 3$ or $x^2 = -1$ (rejected)

$\implies x = \pm\sqrt{3}$.

Next $|x^2 - 1| = 3 \implies x^2 - 1 = 3$ or $x^2 - 1 = -3 \implies x^2 = 4$ or $x^2 = -2$ (rejected)

$\implies x = \pm 2$. Therefore S.S. $= \{-\sqrt{3}, \sqrt{3}, -2, 2\}$

32. The number of solutions to the equation $|2x-1|^3 - 5|2x-1|^2 + 4|2x-1| = 0$ is

(a) 3 (b) 1 ✓(c) 5 (d) 4 (e) 6

Ans: Let $y = |2x-1|$, then the equation will be $y^3 - 5y^2 + 4y = 0 \implies y(y^2 - 5y + 4) = y(y-1)(y-4) = 0 \implies y = 0$, $y = 1$ or $y = 4$. So $|2x-1| = 0 \implies 2x - 1 = 0 \implies x = \frac{1}{2}$, $|2x-1| = 1 \implies 2x - 1 = \pm 1 \implies 2x = 0$ or $2x = 2 \implies x = 0$ or $x = 1$ or finally $|2x-1| = 4 \implies 2x - 1 = \pm 4 \implies 2x = 5$ or $2x = -3 \implies x = \frac{5}{2}$ or $x = -\frac{3}{2}$.

There are 5 solutions and S.S. $= \left\{ \frac{1}{2} , 0 , 1 , \frac{5}{2} , -\frac{3}{2} \right\}$

33. Solve the equation $|2x+8|^2 - |9x+36| - 9 = 0$

Ans: First simplify $|2(x+4)|^2 - |9(x+4)| - 9 = 0 \implies 4|x+4|^2 - 9|x+4| - 9 = 0$. Next let $y = |x+4|$ to get the equation $4y^2 - 9y - 9 = 0 \implies (4y+3)(y-3) = 0 \implies y = -\frac{3}{4}$ or $y = 3 \implies |x+4| = -\frac{3}{4}$ (which is impossible) or $|x+4| = 3 \implies x + 4 = \pm 3 \implies x = -7$ or $x = -1 \implies$ S.S. $= \{-7 , -1\}$

34. Find the sum of all solutions of the equation $(x+3)^{2/3} - 2(x+3)^{1/3} = 3$

Ans: Let $y = (x+3)^{1/3}$, then the equation will be changed to $y^2 - 2y - 3 = 0 \implies (y-3)(y+1) = 0 \implies y = 3$ or $y = -1 \implies (x+3)^{1/3} = 3$ or $(x+3)^{1/3} = -1$ by cubing both sides, we get $x+3 = 27$ or $x+3 = -1 \implies x = 24$ or $x = -4 \implies$ sum of all solutions $= 24 + (-4) = 20$

35. All the solutions of the equation $4x^{-2} - x^{-1} = 5$ lie in the interval:

(a) $[-2, 0]$ (b) $(-3, 0]$ (c) $(-1, 2)$ ✓(d) $[-1, 1)$ (e) $[1, 3)$

Ans: Let $u = x^{-1}$, then the new equation is $4u^2 - u - 5 = 0 \implies (4u-5)(u+1) = 0$
$\implies u = \frac{5}{4}$ or $u = -1 \implies x^{-1} = \frac{5}{4}$ or $x^{-1} = -1 \implies x = \frac{4}{5}$ or $x = -1$. Checking all the above intervals, we can see that $-1 \le -1$, $\frac{4}{5} < 1$. Thus the correct interval is $[-1, 1)$

36. The equation $\sqrt[3]{\sqrt{x^2 + x + 44}} = 2$ has

✓(a) two rational roots (b) two irrational roots (c) no real roots
(d) only one positive real root (e) only one negaitive real root

Ans: $\sqrt[6]{x^2 + x + 44} = 2$ raise to power 6 to get $x^2 + x + 44 = 2^6 = 64 \implies x^2 + x - 20 = 0$
$\implies (x+5)(x-4) = 0 \implies x = -5$ or $x = 4$. We must check, for $x = -5$: $\sqrt[3]{\sqrt{25 - 5 + 44}}$
$= \sqrt[3]{\sqrt{64}} = \sqrt[3]{8} \stackrel{?}{=} 2$ (T). For $x = 4$: $\sqrt[3]{\sqrt{16 + 4 + 44}} = \sqrt[3]{\sqrt{64}} = \sqrt[3]{8} \stackrel{?}{=} 2$ (T).
Therefore, S.S. $= \{-5 , 4\}$, where both roots are rational numbers.

37. Find the solution set of the equation $\sqrt{x} = \sqrt{4 + 7\sqrt{x}} - 2$

Ans: $\sqrt{x} + 2 = \sqrt{4 + 7\sqrt{x}}$. Square both sides to get $x + 4\sqrt{x} + 4 = 4 + 7\sqrt{x} \implies x = 3\sqrt{x}$ $\implies x^2 = 9x \implies x^2 - 9x = 0 \implies x(x-9) = 0 \implies x = 0$ or $x = 9$. As usual, we have to check. For $x = 0: \sqrt{0} \stackrel{?}{=} \sqrt{4} - 2$ (T). For $x = 9: \sqrt{9} \stackrel{?}{=} \sqrt{25} - 2$ (T). The solution set = $\{0\ ,\ 9\}$

38. The solution set of the equation $\sqrt{2x} = \sqrt{x+7} - 1$ consists of

(a) two even integers (b) only one odd integer (c) two odd integers

✔(d) only one even integer (e) one odd and one even integers

Ans: $\sqrt{x+7} = 1 + \sqrt{2x} \implies x + 7 = \left(1 + \sqrt{2x}\right)^2 = 1 + 2\sqrt{2x} + 2x \implies 6 - x = 2\sqrt{2x}$ $\implies \left(6 - x\right)^2 = \left(2\sqrt{2x}\right)^2 \implies 36 - 12x + x^2 = 8x \implies x^2 - 20x + 36 = 0 \implies (x-18)(x-2) = 0 \implies x = 18$ or $x = 2$. We must check, for $x = 18: \sqrt{36} \stackrel{?}{=} \sqrt{25} - 1$ (F), so 18 is rejected. For $x = 2: \sqrt{4} \stackrel{?}{=} \sqrt{9} - 1$ (T) $\implies$ S.S. = $\{2\}$ which is an even integer.

39. The product of all solutions of the equation $\sqrt[3]{x^4} - 2\sqrt[3]{x^2} - 3 = 0$ is

(a) -9 (b) 27 (c) -16 (d) 16 ✔(e) -27

Ans: $x^{4/3} - 2\,x^{2/3} - 3 = 0$. Let $u = x^{2/3}$, so the equation is $u^2 - 2u - 3 = 0 \implies (u-3)(u+1)$ $= 0 \implies u = 3$ or $u = -1 \implies x^{2/3} = 3 \implies x^2 = 3^3 = 27 \implies x = \pm\sqrt{27} = \pm 3\sqrt{3}$
or $x^{2/3} = -1$ which is impossible, because $x^{2/3} = (\sqrt[3]{x})^2 \geq 0$.
Therefore the product of all solutions = $(-3\sqrt{3})(3\sqrt{3}) = -27$

40. The number of real solutions to the equation $\sqrt[3]{x-1} - \sqrt{x+1} = 0$ is

(a) 3 (b) 2 ✔(c) 0 (d) 1 (e) 4

Ans: $\sqrt[3]{x-1} = \sqrt{x+1}$. Raise both sides to power 6 to get $(x-1)^2 = (x+1)^3 \implies x^2 - 2x + 1 = x^3 + 3x^2 + 3x + 1 \implies x^3 + 2x^2 + 5x = 0 \implies x(x^2 + 2x + 5) = 0 \implies x = 0$ or $x^2 + 2x + 5 = 0$, which has no real roots for $b^2 - 4ac = 4 - 20 = -16 < 0$. Therefore the only possible real root is 0. Even that we must check, since we raised to an even power. For $x = 0: \sqrt[3]{-1} - \sqrt{1} \stackrel{?}{=} 0$ (F). Thus the equation has no real solution.

41. Find the solution set of the equation $\sqrt{x} - \sqrt[4]{x} = 2$

Ans: Let $u = \sqrt[4]{x}$, so $\sqrt{x} = u^2$. The equation turns to be $u^2 - u - 2 = 0 \implies (u-2)(u+1) = 0 \implies u = 2$ or $u = -1 \implies \sqrt[4]{x} = 2 \implies x = 2^4 = 16$ or $\sqrt[4]{x} = -1$ (rejected, because $\sqrt[4]{x} \geq 0$) $\implies$ S.S. $= \{16\}$

42. Find the sum of all solutions of the equation $\sqrt{x^3 - 6x} + x = 0$

Ans: Seperate the radical to get $\sqrt{x^3 - 6x} = -x \implies x^3 - 6x = (-x)^2 = x^2 \implies x^3 - x^2 - 6x = 0 \implies x(x^2 - x - 6) = 0 \implies x(x-3)(x+2) = 0 \implies x = 0\ ,\ 3\ ,$ or -2. For $x = 0: \sqrt{0} + 0 \stackrel{?}{=} 0$ (T). For $x = 3: \sqrt{9} + 3 \stackrel{?}{=} 0$ (F). For $x = -2: \sqrt{4} + (-2) \stackrel{?}{=} 0$ (T).

Thus the sum of all solutions $= 0 + (-2) = -2$

43. Solve the following inequalities: (i) $|5x - 10| \geq 0$ (ii) $|3x + 12| > 0$ (iii) $|2x - 4| < 0$ (iv) $|2x + 1| \leq 0$ (v) $|-5x - 2| \leq -4$ (vi) $|3x + 2| \geq -1$

Ans: (i) This is always true, so S.S. $= \Re = (-\infty, \infty)$ (ii) This one is always true except if $3x + 12 = 0 \implies$ S.S. $= \Re - \{-4\} = (-\infty, -4) \cup (-4, \infty)$ (iii) It is never true, so S.S. $= \phi$ (iv) It is true only when $2x + 1 = 0 \implies$ S.S $= \left\{-\frac{1}{2}\right\}$ (v) Again is never true, so S.S. $= \phi$ (vi) While the last one is always true which means that S.S. $= \Re = (-\infty, \infty)$

44. Find the solution set, in interval notation, of the compound inequality $\frac{7x+6}{6} > \frac{x+2}{2}$ or $4(x+4) > 2(2-x)$

Ans: $7x + 6 > 3(x+2)$ or $4x + 16 > 4 - 2x \implies 7x - 3x > 6 - 6$ or $4x + 2x > 4 - 16 \implies 4x > 0$ or $6x > -12 \implies x > 0$ or $x > -3 \implies$ S.S. $= (0, \infty) \cup (-3, \infty) = (-3, \infty)$

45. Solve the inequality $-2(1-2x)^2 + 8 \geq 0$

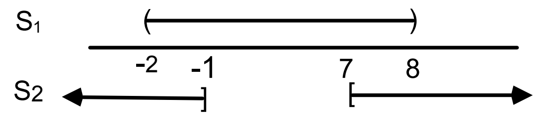
Ans: In this problem, I am going to use the idea: For $a > 0$, $x^2 \leq a^2$ is equivalent to $|x| \leq a$ and may be later we use $x^2 \geq a^2$ is equivalent to $|x| \geq a$. Back to the problem

$-2(1-2x)^2 \geq -8 \implies (1-2x)^2 \leq 4 \implies |1 - 2x| \leq 2 \implies |2x - 1| \leq 2 \implies$

$-2 \leq 2x - 1 \leq 2 \implies -1 \leq 2x \leq 3 \implies -\frac{1}{2} \leq x \leq \frac{3}{2} \implies$ S.S. $= \left[-\frac{1}{2}, \frac{3}{2}\right]$

46. Solve the the inequality $-4 < -2|x-3| + 6 \leq -2$

Ans: $\Longrightarrow -10 < -2|x-3| \leq -8 \Longrightarrow 5 > |x-3| \geq 4$.
Now we seperate the inequality into $|x-3| < 5$ and $|x-3| \geq 4 \Longrightarrow \left(-5 < x-3 < 5\right)$ and $\left(x-3 \leq -4 \text{ or } x-3 \geq 4\right) \Longrightarrow \left(-2 < x < 8\right)$ and $\left(x \leq -1 \text{ or } x \geq 7\right) \Longrightarrow$ S.S $= S_1 \cap S_2 = (-2,-1] \cup [7,8)$

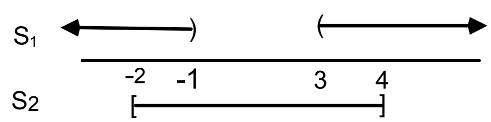


47. Solve the the inequality $3 < x^2 - 2x \leq 8$

Ans: It can be done by seperating the inequality into $x^2 - 2x > 3$ and $x^2 - 2x \leq 8$
$\Longrightarrow x^2 - 2x - 3 > 0$ and $x^2 - 2x - 8 \leq 0 \Longrightarrow (x+1)(x-3) > 0$ and $(x+2)(x-4) \leq 0$.
We solve each inequality by using the sign test, then we find the intersection of both solutions.

	$x < -1$	$-1 < x < 3$	$x > 3$
(x+1)	- -	+ +	+ +
(x-3)	- -	- -	+ +
L.S. is :	+	-	+

	$x < -2$	$-2 < x < 4$	$x > 4$
(x+2)	- -	+ +	+ +
(x-4)	- -	- -	+ +
L.S. is :	+	-	+



$S_1 = (-\infty,-1) \cup (3,\infty)$, $S_2 = [-2,4]$, so S.S $= S_1 \cap S_2 = [-2,-1) \cup (3,4]$

48. Solve the inequality $\dfrac{-(3-x)^2\,(x+1)^3}{(x+2)^4\,(4-x)} \geq 0$

Ans: Multiply by (-1) to get $\dfrac{(3-x)^2\,(x+1)^3}{(x+2)^4\,(4-x)} \leq 0$.
Notice that $x \neq -2$, $x \neq 4$ while $x = 3$ and $x = -1$ should be included in the solution set.
By sign test S.S. $(-\infty,-2) \cup (-2,-1] \cup (4,\infty) \cup \{3\}$

	$x < -2$	$-2 < x < -1$	$-1 < x < 3$	$3 < x < 4$	$x > 4$
$(3-x)^2$	+ +	+ +	+ +	+ +	+ +
$(x+1)^3$	- -	- -	+ +	+ +	+ +
$(x+2)^4$	+ +	+ +	+ +	+ +	+ +
(4-x)	+ +	+ +	+ +	+ +	- -
L.S. is :	-	-	+	+	-

49. Solve the inequality $|x-1| \geq |2x-3|$

Ans: Since both sides are positive, then one way to solve this inequality is by squaring both sides, so $|x-1|^2 \geq |2x-3|^2 \Longrightarrow (x-1)^2 \geq (2x-3)^2 \Longrightarrow x^2 - 2x + 1 \geq 4x^2 - 12x + 9 \Longrightarrow 3x^2 - 10x + 8 \leq 0 \Longrightarrow (3x-4)(x-2) \leq 0$.

By sign test S.S. $= \left[\dfrac{4}{3}, 2\right]$

	$x < \frac{4}{3}$	$\frac{4}{3} < x < 2$	$x > 2$
(3x-4)	- -	+ +	+ +
(x-2)	- -	- -	+ +
L.S. is :	+	-	+

50. Solve the inequality $\dfrac{x-2}{x-1} \le \dfrac{x+3}{x+1}$

Ans: $\dfrac{x-2}{x-1} - \dfrac{x+3}{x+1} \le 0$

$\implies \dfrac{(x^2-x-2)-(x^2+2x-3)}{(x-1)(x+1)} \le 0$

$\implies \dfrac{1-3x}{(x-1)(x+1)} \le 0 \;, x \ne \pm 1.$

Again by sign test S.S. $= \left(-1, \dfrac{1}{3}\right] \cup \left(1, \infty\right)$

(1-3x)	+ +		+ +		− −		− −
(x-1)	− −		− −		− −		+ +
(x+1)	− −		+ +		+ +		+ +
L.S. is :	+	-1	−	$\frac{1}{3}$	+	1	−

51. Find the values of k for which the equation $kx^2 + 2kx + 8x + 25 = 0$ has
(a) **two distinct real** roots (b) real roots

Ans: (a) The equation $kx^2 + 2(k+4)x + 25 = 0$ has two distinct real roots when the discriminant $= b^2 - 4ac > 0$, where $a = k$, $b = 2(k+4)$, $c = 25$
$\implies 4(k+4)^2 - 4k(25) > 0 \implies (k+4)^2 - 25k > 0$
$\implies k^2 - 17k + 16 > 0 \implies (k-1)(k-16) > 0$

By sign test $k \in (-\infty, 1) \cup (16, \infty)$

(k -1)	− −		+ +		+ +
(k -16)	− −		− −		+ +
L.S. is :	+	1	−	16	+

(b) saying **real** roots here means the roots may be two equal real roots or two distinct real roots. In this case, $b^2 - 4ac = 0$ or $b^2 - 4ac > 0 \implies b^2 - 4ac \ge 0$ (see part a)
$\implies (k-1)(k-16) \ge 0 \implies k \in (-\infty, 1] \cup [16, \infty)$

52. Find the values of k for which the equation $x^2 - kx + k = -3$ has **no real** roots.

Ans: Saying the equation $x^2 - kx + (k+3) = 0$ has no real roots as saying it has two distinct non-real conjugate complex roots. The discriminant $= b^2 - 4ac < 0$
$\implies k^2 - 4(k+3) < 0 \implies k^2 - 4k - 12 < 0$
$\implies (k-6)(k+2) < 0.$

From the adjacent figure $k \in (-2, 6)$.

(k -6)	− −		− −		+ +
(k +2)	− −		+ +		+ +
L.S. is :	+	-2	−	6	+

Mathematics is not just reading ....

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