Math 001 – Real Numbers – Done By : A. Al-Shallali

Let us denote some important sets by the following letters:

N = The set of **natural** numbers (الأعداد الطبيعية) $= \{1, 2, 3, 4, ...\}$
W = The set of **whole** numbers (الأعداد الكلية) $= \{0, 1, 2, 3, ...\}$
Z = The set of **integer** numbers (الأعداد الصحيحة) $= \{..., -2, -1, 0, 1, 2, ...\}$
Q = The set of **rational** numbers (الأعداد النسبية) $= \left\{ x | x = \dfrac{p}{q};\ p, q \in \mathrm{Z}\ ,\ q \neq 0 \right\}$
H = The set of **irrational** numbers (الأعداد الغير نسبية) $= \{x | x \text{ is a real number and } x \notin \mathrm{Q}\}$
$\mathfrak{R}$ = The set of **real** numbers (الأعداد الحقيقية) $= \{x | x \in \mathrm{Q} \text{ or } x \in \mathrm{H}\}$

Notice that any (a) **integer** , (b) **fraction** (كسر إعتيادي) , (c) **terminating decimal** (كسر عشري منتهي) or (d) **repeating decimal** (كسر عشري دوري) number can be written in the form $\dfrac{p}{q}$, where , $p, q \in Z, q \neq 0$, therefore it is a **rational** number.

While any real number that does not belong to any of the above four parts is an **irrational** number.

For example : $0 = \dfrac{0}{1}$, $-3 = \dfrac{-3}{1}$, $-\dfrac{4}{5}$, $1.23 = \dfrac{123}{100}$, $0.059 = \dfrac{59}{1000}$, $1.3333... = 1.\overline{3} = \dfrac{4}{3}$, and $0.262626... = 0.\overline{26} = \dfrac{26}{99}$ all are **rational** numbers . The numbers π , $\dfrac{2\pi}{5}$, $\dfrac{\sqrt{8}}{3}$,$\sqrt[3]{2}$ and 5.204781.....all are **irrational** numbers , while , $\dfrac{7}{0}$, $\sqrt{-9}$ and $\sqrt[4]{-16}$ are **not real** numbers.

Moreover $\dfrac{22}{7}$ and $3.14 = \dfrac{314}{100}$ are **rational** numbers while π is an **irrational** number.This is because $\pi \neq 3.14$ or $\dfrac{22}{7}$ but $\pi = 3.1415926536\ldots\ldots$ (non-terminating and non-repeating decimal) (كسر عشري غير منتهي و غير دوري) . So $\pi \approx \dfrac{22}{7}$ or 3.14. In fact $\boxed{3.14 < \pi < \dfrac{22}{7}}$

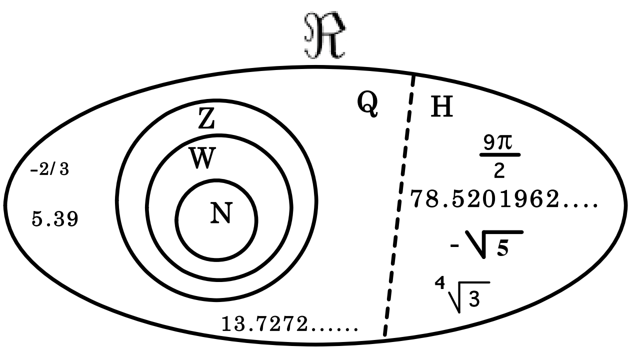
Problem : Show that $3.\overline{2}$ and $51.\overline{43}$ are rational numbers.

Proof : a)
$$\begin{array}{rrcl} \text{Let} & x & = & 3.222\ldots \\ \Longrightarrow & 10x & = & 32.222\ldots \\ \hline \end{array}$$
subtract to get: $9x = 29 \Longrightarrow x = \dfrac{29}{9} \in \mathrm{Q}$

b)
$$\begin{array}{rrcl} \text{Let} & x & = & 51.4343\ldots \\ \Longrightarrow & 100x & = & 5143.4343\ldots \\ \hline \end{array}$$
subtract to get: $99x = 5092 \Longrightarrow x = \dfrac{5092}{99} \in \mathrm{Q}$

• **Notice** that : (1) $\mathrm{N} \subset \mathrm{W} \subset \mathrm{Z} \subset \mathrm{Q} \subset \mathfrak{R}$ and $\mathrm{H} \subset \mathfrak{R}$ (2) $\mathrm{Q} \bigcup \mathrm{H} = \mathfrak{R}$ **and** $\mathrm{Q} \bigcap \mathrm{H} = \emptyset$

All this can be illustrated by the following figure:



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Properties of real numbers:

Let a , b , and c be real numbers		
Property Name	**For Addition**	**For Multiplication**
Closure (الاغلاق)	$a+b$ is a **unique real** number	$a \cdot b$ is a **unique real** number
Commutative (التبادل)	$a+b=b+a$	$a \cdot b = b \cdot a$
Associative (التجميع)	$(a+b)+c=a+(b+c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Identity (العنصر المحايد)	$a+0=0+a=a$ 0 is called the additive identity	$a \cdot 1 = 1 \cdot a = a$ 1 is called the multiplicative identity
Inverse (المعكوس)	$a+(-a)=(-a)+a=0$ $-a$ is called the additive inverse of a	$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1 \ , \ a \neq 0$ $\frac{1}{a}$ is called the multiplicative inverse of a
Distributive (التوزيع)	$a \cdot (b+c) = a \cdot b + a \cdot c$	

a set A is said to be **closed** under an operation $*$ if $a, b \in$ A, then $a * b \in$ A. It can be seen that:

- The sets N , W , Z, Q and $\Re$ all are **closed** under **addition** and **multiplication**.
- Only the sets Z , Q and $\Re$ are **closed** under **subtraction** , e.g. , 2 , 5 $\in$ N but $2-5=-3 \notin$ N.
- None of the above sets is **closed** under **division** , e.g. , 4 , 7 $\in$ N but $\frac{4}{7} \notin$ N ,similarly for W and Z . Also 0.5 , 0 $\in$ Q, $\Re$ but $\frac{0.5}{0} \notin$ Q, $\Re$.
- The set H is **not closed** under any operation , e.g. ,$\sqrt{3}$, $-\sqrt{3} \in$ H but $\sqrt{3}+(-\sqrt{3})=0 \notin$ H ; $\sqrt{3}-\sqrt{3}=0 \notin$ H; $\sqrt{3} \,.\, \sqrt{3}=3 \notin$ H and $\frac{\sqrt{3}}{\sqrt{3}}=1 \notin$ H.

Let us summarize all of this in the following table:

Set	$+$	$-$	$\times$	$\div$
N	yes	no	yes	no
W	yes	no	yes	no
Z	yes	yes	yes	no
Q	yes	yes	yes	no
H	no	no	no	no
$\Re$	yes	yes	yes	no
$\Re^*$	yes	yes	yes	yes

$\Re^* = \Re - \{0\}$ = set of **nonzero** real numbers

Example : The set of perfect squares = S = $\{1^2, 2^2, 3^2, 4^2, 5^2, 6^2, \ldots\} = \{1, 4, 9, 16, 25, 36, \ldots\}$ is not closed under addition, e.g. , $1, 4 \in$ S but $1+4=5 \notin$ S. This set is also not closed under subtraction or division, it is closed under multiplication only

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Example : The set $A = \{-1, 0\}$ is **not** closed under addition for : $(-1)+(-1) = -2 \notin A$, **not** closed under subtraction for : $0-(-1) = 1 \notin A$, **not** closed under multiplication for : $(-1)(-1) = 1 \notin A$ and **not** closed under division for : $\dfrac{-1}{-1} = 1 \notin A$.

Notes:

- All the sets N , W , Z , Q , H and $\Re$ satisfy the **commutative** property of addition and multiplication , i . e . , $a+b = b+a$ & $ab = ba$, whenever a, b are in the set.
- All the above sets satisfy the **associative** property of addition and multiplication , i . e . , $(a+b)+c = a+(b+c)$ & $(ab)c = a(bc)$, whenever a, b, c are in the set.
- The sets $\Re$, Q , Z and W contain the **identity** element of addition (0) and the **identity** element of multiplication (1). The set N contains only the **identity** element of multiplication while the set H **does not** contain any one of them.

The table below shows some additive and multiplicative inverses of some real numbers:

Number	Additive Inverse	Multiplicative Inverse
-1	1	-1
0	0	does not exist
$-\pi/2$	$\pi/2$	$-2/\pi$
$2-\sqrt{3}$	$\sqrt{3}-2$	$1/(2-\sqrt{3}) = 2+\sqrt{3}$
-0.04	0.04	-25
$-3\,\frac{2}{5}$	$3\,\frac{2}{5} = \frac{17}{5}$	$-\frac{5}{17}$

Problem : Name the property in each of the following statements :

1) $7+[5+(-5)] = 7+[(-5)+5]$: commutative property of addition
2) $6 \cdot 1 = 6$: identity property of multiplication
3) $2a+0 = 2a$: identity property of addition
4) $9 \cdot \left(\dfrac{1}{9}\right) = \left(\dfrac{1}{9}\right).9$: commutative property of multiplication
5) $3c+(-3c) = 0$: inverse property of addition
6) $3x+18 = 3(x+6)$: distributive property
7) $[2+(-2)]+0 = 2+[(-2)+0]$: associative property of addition
8) $\left(\dfrac{3}{2}\right) \cdot \left(\dfrac{2}{3}\right) = 1$: inverse property of multiplication
9) $\left(2\right) \cdot \left(\dfrac{1}{2}\right) + 0 = 1$: inverse prop. of multiplication and identity prop. of addition
10) $2(a+b) = 2b+2a$: distributive property and commutative property of addition

Problem : Tell to which sets each of the following numbers belongs:

Numer	**N**	**W**	**Z**	**Q**	**H**	$\mathfrak{R}$	**Note**
$\dfrac{-3/4}{-2/3}$	X	X	X	$\surd$	X	$\surd$	$\dfrac{-3/4}{-2/3} = \dfrac{9}{8}$
$\dfrac{4-4}{\pi}$	X	$\surd$	$\surd$	$\surd$	X	$\surd$	$\dfrac{4-4}{\pi} = \dfrac{0}{\pi} = 0$
$-\dfrac{0.9}{0.03}$	X	X	$\surd$	$\surd$	X	$\surd$	$-\dfrac{0.9}{0.03} = -\dfrac{9/10}{3/100} = -30$
$\dfrac{\sqrt{8}}{-4}$	X	X	X	X	$\surd$	$\surd$	$\dfrac{\sqrt{8}}{-4} = \dfrac{2\sqrt{2}}{-4} = -\dfrac{\sqrt{2}}{2}$
$\dfrac{\sqrt{16\pi^2}}{2\pi}$	$\surd$	$\surd$	$\surd$	$\surd$	X	$\surd$	$\dfrac{\sqrt{16\pi^2}}{2\pi} = \dfrac{4\pi}{2\pi} = 2$
$(3.\overline{25})^2$	X	X	X	$\surd$	X	$\surd$	$= (3.\overline{25})(3.\overline{25})$ product of 2 rational no. is rationl
$\dfrac{\sqrt[3]{0.001}}{\sqrt{0.09}}$	X	X	X	$\surd$	X	$\surd$	$\dfrac{\sqrt[3]{0.001}}{\sqrt{0.09}} = \dfrac{0.1}{0.3} = \dfrac{1}{3}$
$\dfrac{6-4}{2-2}$	X	X	X	X	X	X	$\dfrac{6-4}{2-2} = \dfrac{2}{0}$ is undefined
$\dfrac{\sqrt{\pi^3}}{\sqrt{4\pi}}$	X	X	X	X	$\surd$	$\surd$	$\dfrac{\sqrt{\pi^3}}{\sqrt{4\pi}} = \dfrac{\pi\sqrt{\pi}}{2\sqrt{\pi}} = \dfrac{\pi}{2}$
$7.23234235\ldots\ldots$	X	X	X	X	$\surd$	$\surd$	non-terminating and non-repeating decimal
$\sqrt{-4}\;\sqrt[3]{-1}$	X	X	X	X	X	X	$= 2\sqrt{-1}\,(-1) = -2\sqrt{-1}$ is an imaginary number

Order of Operations:

Grouping Symbols are parentheses ( ) , braces {} , and brackets [ ]

To evaluate a complex expression, you need to get rid of all grouping symbols (if present) by performing the operations within the grouping symbols, innermost grouping symbols first, while observing the order given in steps 1 to 3.

step 1 Evaluate exponents expressions.
step 2 Do multiplication and division as they occur from left to right.
step 3 Do addition and subtraction as they occur from left to right.

Example: Evaluate the expression $2[3-\{1+4(1-6)\}]$

Solution: $\underbrace{2\,[3-\underbrace{\{1+4\,\underbrace{(1-6)}_{(1)}\}}_{(2)}]}_{(3)} = 2[3-\{1-4(-3)\}] = 2[3-\{1+12\}] = 2[3-13] = 2[-10] = -20$

Example: Evaluate the expression $\dfrac{2[2^3-1]+4\cdot 5-8\div 2}{2\cdot 3^2-6\cdot 2-3\div 3}$

Solution: $= \dfrac{2[8-1]+4\cdot 5-8\div 2}{2\cdot 9-6\cdot 2-3\div 3} = \dfrac{2\cdot 7+4\cdot 5-8\div 2}{18-12-1} = \dfrac{14+20-4}{5} = \dfrac{30}{5} = 6$

Problem: Determine whether each of the following statements is true (T) or false (F) and explain your answer:

Statement	T/F	Explanation
the additive inverse of the number $(1-2\sqrt{2})(1+2\sqrt{2})$ is 7	T	$(1-2\sqrt{2})(1+2\sqrt{2}) = 1-8 = -7$, so the additive inverse $= 7$
the set $\mathrm{A} = \{1, 2, \frac{1}{2}\}$ satisfies the inverse property of multiplication	T	$\frac{1}{1} = 1 \in \mathrm{A}$, $\frac{1}{2} \in \mathrm{A}$, $\frac{1}{(1/2)} = 2 \in \mathrm{A}$
the set of irrational numbers (H) is closed under addition	F	$\pi, -\pi \in \mathrm{H}$, but $\pi + (-\pi) = 0 \notin \mathrm{H}$
the multiplicative inverse of $5\frac{2}{7}$ is $-\frac{7}{10}$	F	$5\frac{2}{7} = \frac{37}{7}$, so its mult. inverse is $\frac{7}{37}$
any rational number has a multiplication inverse.	F	$0 \in \mathrm{Q}$, but $\frac{1}{0} \notin \mathrm{Q}$
$xy + z = z + yx$ is true because of commutative property of addition and multiplication	T	$xy + z = z + xy = z + yx$
set of perfect squares is closed under multiplication.	T	a^2, b^2 are two perfect squares, then $a^2 \cdot b^2 = (ab)^2$ is also a perfect square
set of real numbers satisfies the associative property of division	F	$(6\div 2)\div 3 = 3\div 3 = 1$, but $6\div(2\div 3) = 6\cdot\frac{3}{2} = 9$
the difference of any two integers is a rational number	T	the difference of any two integers is an integer, so it is a rational number
both sets of natural and whole numbers contain an addition identity	F	$0 \notin \mathrm{N}$
every even integer has an additive inverse	T	the additive inverse of $2n$ is $-2n$ which's even, where $n \in \mathrm{Z}$
the interval $(-3, 0) \cap (-1, 1)$ contains two integers	F	in fact $(-3,0)\cap(-1,1) = (-1,0)$ which contains no integer at all
$(\mathrm{N}\cap\mathrm{W}) \cup (\mathrm{W}\cap\mathrm{Z}) \cup (\mathrm{Q}\cap\mathrm{H}) = \mathrm{N}$	F	$= \mathrm{N} \cup \mathrm{W} \cup \phi = \mathrm{W} \cup \phi = W$
$\left(-\frac{5}{7}\right)\left(\frac{2\pi}{3}\right)$ is a real number	T	closure property of multiplication of real numbers
$\sqrt{2}\,[\frac{\sqrt{2}-1}{2\sqrt{2}} - \frac{1}{2}]$ is a rtional number	T	$= \sqrt{2}\,[\frac{\sqrt{2}-1-\sqrt{2}}{2\sqrt{2}}] = \sqrt{2}\,[\frac{-1}{2\sqrt{2}}] = -\frac{1}{2}$ is rational