Math 001 – Notes & Examples On Rational Functions

<u>Def.</u>: If P(x) and Q(x) are polynomials with $Q(x) \neq 0$, then $f(x) = \dfrac{P(x)}{Q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$ is called a rational function.

<u>x–intercept(s)</u>: To find the x–intercept(s), first factor P(x) and Q(x) and then cancel the common factors. Set the new numerator equal to 0 and solve for x. For example: $f(x) = \dfrac{x^2 - 2x - 3}{x^2 - 5x - 6} = \dfrac{(x+1)(x-3)}{(x+1)(x-6)} = \dfrac{x-3}{x-6}$, $x \neq -1$, $x \neq 6$. Let $x - 3 = 0 \implies x = 3 \implies$ x–intercept $= (3, 0)$ only.

<u>Hole</u>: The graph of $f(x)$ has a hole at $x = a$ if $(x-a)$ is a common factor of P(x) and Q(x) and a will not be a zero of the (simplified) new denominator anymore,i.e., $y = \dfrac{P(x)}{Q(x)} = \dfrac{(x-a)N(x)}{(x-a)D(x)} = \dfrac{N(x)}{D(x)}$, $D(a) \neq 0$. The hole (missing point) $= \left(a, \dfrac{N(a)}{D(a)}\right)$. For example, the function $y = \dfrac{(x-1)(x+5)}{(x-1)(x-3)} = \dfrac{x+5}{x-3}$, $x \neq 1$, $x \neq 3$ has a hole at the point $(1, -3)$.

<u>Vertical Asymptote</u>: The vertical line $x = a$ is called a vertical asymptote of the rational function $f(x)$ if $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a^-$ or $x \to a^+$ (graph is closed to the the line $x = a$)
In other words $x = a$ is a vertical asymptote of the function $y = \dfrac{P(x)}{Q(x)}$ if $Q(a) = 0$ and $P(a) \neq 0$. To find these lines: Factor P(x) and Q(x), cancel the common factors, then set the new denominator equal to 0 and solve for x. For example $y = \dfrac{x^2 + 2x - 3}{x^2 - 1} = \dfrac{(x-1)(x+3)}{(x-1)(x+1)} = \dfrac{x+3}{x+1}$, $x \neq -1$, $x \neq 1$ has a vertical asymptote $x = -1$ (hole $= (1, 2)$).

<u>Horizental Asymptote</u>: The horizontal line $y = b$ is called a horizontal asymptote of the graph of a rational function $f(x)$ if $f(x) \to b$ (graph is closed to the the line $y = b$) as $x \to -\infty$ or $x \to \infty$.
One of the following may occur: (a) If degree of $P(x) = n <$ degree of $Q(x) = m$, then $y = 0$ is a horizontal asymptote. (b) If $n = m$, then $y = \dfrac{a_n}{b_m}$ is a horizontal asymptote. (c) If $n > m$, then there is no horizontal asymptote. For example the horizontal asymptote of the functions $y = \dfrac{2x - 1}{x^3 - 5x^2 + 3}$ and $y = \dfrac{x^2 - 1}{2x^3 + x^2 + 2}$ is $y = 0$. The horizontal asymptote of $y = \dfrac{1 + 2x - 3x^2}{4x^2 - 5x - 7}$ is $y = -\dfrac{3}{4}$

<u>Slant Asymptote</u>: The slant line $y = ax + b$ is called a slant asymptote of the graph of a rational function $f(x)$ if $f(x) \to ax + b$ (graph is closed to the the line $y = ax + b$) as $x \to -\infty$ or $x \to \infty$.
The rational function $f(x) = \dfrac{P(x)}{Q(x)}$ has a slant asymptote when the degree of P(x) is one greater than the degree of Q(x). To find it, just divide P(x) by Q(x) to get a quotient of the form $ax + b$, $a \neq 0$, the slant asymptote is $y = ax + b$. In fact $f(x) = \dfrac{P(x)}{Q(x)} = (ax + b) + \dfrac{R(x)}{Q(x)}$, where degree of R(x) < degree of Q(x). So as $x \to \pm\infty$, $\dfrac{R(x)}{Q(x)} \to 0$ and therefore $f(x) \to ax + b$. For example, the slant asymptote of $f(x) = \dfrac{x^3 + 3x^2 + 1}{x^2 + 1} = (x + 3) + \dfrac{-x - 2}{x^2 + 1}$ is $y = x + 3$.
See the next table for more examples and notes.

Table: Intercepts, Holes, and Asymptotes of Rational Functions

#	Function: $y =$	x–int.	y–int.	Hole	V. Asym.	H. Asym.	S. Asym.
1	$\dfrac{x-1}{(x+1)(x-3)}$	$(1,0)$	$\left(0,\dfrac{1}{3}\right)$	—	$x=-1\,,\,x=3$	$y=0$	—
2	$\dfrac{2x^2+4}{(x-3)(x+2)}$	—	$\left(0,-\dfrac{2}{3}\right)$	—	$x=3\,,\,x=-2$	$y=2$	—
3	$\dfrac{(x+1)^2}{x(x-2)}$	$(-1,0)$	—	—	$x=0\,,\,x=2$	$y=1$	—
4	$\dfrac{(x-2)(x+2)}{x^4+1}$	$(-2,0)\,,\,(2,0)$	(0,-4)	—	—	$y=0$	—
5	$\dfrac{2x\,\cancel{(x-1)}^{\,-1}}{(1+x)\,\cancel{(1-x)}}$	$(0,0)$	(0,0)	(1,-1)	$x=-1$	$y=-2$	—
6	$\dfrac{\cancel{x}\,(x-3)}{\cancel{x}\,(x+1)}$	$(3,0)$	—	$(0,-3)$	$x=-1$	$y=1$	—
7	$\dfrac{4(x-1)^{\cancel{2}}}{\cancel{(x-1)}\,(x+2)}$	—	$(0,-2)$	$(1,0)$	$x=-2$	$y=4$	—
8	$\dfrac{\cancel{(x-2)}\,(x-3)}{(x-2)^{\cancel{2}}}$	$(3,0)$	$\left(0,\dfrac{3}{2}\right)$	—	$x=2$	$y=1$	—
9	$\dfrac{2x^3}{x^2+1}$	$(0,0)$	$(0,0)$	—	—	—	$y=2x$
10	$\dfrac{3x\,\cancel{(x-1)}\,(x+3)}{\cancel{(x-1)}\,(x+4)}$	$(0,0)\,,(-3,0)$	$(0,0)$	$\left(1,\dfrac{12}{5}\right)$	$x=-4$	—	$y=3x-3$
11	$\dfrac{(x+1)^{\cancel{3}}}{\cancel{(x+1)^2}}=x+1$	—	$(0,1)$	$(-1,0)$	—	—	—

Notice that in function # :

(6) The point $(0,-3)$ lies on the y–axis, but it's not a y–intercept. It's a hole point, because $x \neq 0$

(7) The point $(1,0)$ lies on the x–axis, but it's not an x–intercept. Again it's a hole point, because $x \neq 1$

(8) At $x=2$, there's no hole point. In fact the line $x=2$ is a vertical asymptote, because $(x-2)$ is still a factor of the denominator.

(11) The graph of this function is just the slant line $y=x+1$ with a hole at $(-1,0)$.

Note1: The domain of all of the above functions is the set of all values of x for which the **original** denominator is not equal to 0. For example, the domain of function #7 is $\Re - \{-2,1\} = (-\infty,-2) \cup (-2,1) \cup (1,\infty)$

Note2: The graph of a rational function never intersects its vertical asymptotes. But it may intersect its horizontal or slant asymptotes. In function #2, the graph intersects its horizontal asymptote at the point $(-8,2)$ [by solving $f(x)=2$] while in function #6, it doesn't [check by solving $f(x)=1$]. Also in function #9, the graph intersects its slant asymptote at the point $(0,0)$ [by solving $f(x)=2x$] while in function #10, it doesn't [check by solving $f(x)=3x-3$].

Done By: A. Al-Shallali