Math 001 – Domain & Range By: A. Al-Shallali

To find the domain and range of a relation or function we have to keep in mind some important notes such as :

- First The **Domain** : The domain of a function $y = f(x)$ is the set of all possible real values of x for which the function is defined .
  Remember that all denominators should not equal to 0 and what is inside the **even** nth root should be non-negative .
  Now let us find the domain of the following relations or functions :

  1. For $x = -2$: The domain $= \{-2\}$
  2. For $y = 3$; $y = 2x + 5$; $y = x^2 - 4$; $y = |x - 2|$; $y = \sqrt[3]{x+4}$; $y = \dfrac{1}{x^2+9}$ and $y = \sqrt{x^2+1}$: The domain$= \Re = (-\infty, \infty)$.
  3. For $y = \dfrac{1}{x^2 - 2x - 3}$: We have $x^2 - 2x - 3 \neq 0 \Rightarrow (x-3)(x+1) \neq 0 \Longrightarrow x \neq 3, -1$.
     Therfore the domain $= (-\infty, -1) \cup (-1, 3) \cup (3, \infty)$.
  4. For $y = \sqrt{x^2 - 5x + 4}$: We have $x^2 - 5x + 4 = (x-4)(x-1) \geq 0 \Longrightarrow$ (by sign test) $\boxed{-\infty < x \leq 1}$ or $\boxed{4 \leq x < \infty}$. So the domain $= (-\infty, 1] \cup [4, \infty)$.
  5. For $y = \dfrac{1}{\sqrt{9 - x^2}}$: Here $9 - x^2 > 0 \Longrightarrow x^2 < 9 \Longrightarrow |x| < 3 \Longrightarrow -3 < x < 3$.
     So the domain $= (-3, 3)$.
  6. For $y = \dfrac{1}{\sqrt[3]{x^2 - x - 2}}$: Here $x^2 - x - 2$ can be negative but **not** 0 , i . e . ,
     $x^2 - x - 2 = (x-2)(x+1) \neq 0 \Longrightarrow x \neq 2, -1 \Longrightarrow$ domain $= (-\infty, -1) \cup (-1, 2) \cup (2, \infty)$
  7. For $y = \sqrt[4]{3 - |x-2|}$: In this case $3 - |x-2| \geq 0 \Longrightarrow |x-2| \leq 3$
     $\Longrightarrow -3 \leq x - 2 \leq 3 \Longrightarrow -1 \leq x \leq 5 \Longrightarrow$ domain $= [-1, 5]$.
  8. Notice that the domain of $f(x) = \sqrt{\dfrac{x+1}{x-3}}$ is not the same as the domain of $g(x) = \dfrac{\sqrt{x+1}}{\sqrt{x-3}}$. The reason for this is that in $f(x)$: $\dfrac{x+1}{x-3} \geq 0$ while in $g(x)$: $(x+1) \geq 0$ and $(x-3) > 0$. In fact $D_f = (-\infty, -1] \cup (3, \infty)$ while $D_g = (3, \infty)$.

- Second The **Range** : The range of $y = f(x)$ is the set of all possible real values of y .
  We have to remember for example that : $|x| \geq 0$; $x^2 \geq 0$; $x^4 \geq 0$ for any $x \in \Re$ and $\sqrt{x} \geq 0$; $\sqrt[4]{x} \geq 0$ for any $x \geq 0$. This implies that $|x| - 2 \geq -2$; $x^2 + 5 \geq 5$; $x^4 - 8 \geq -8$ and $\sqrt{x} - 3 \geq -3$.

  Similarly $-|x| \leq 0$; $-x^2 \leq 0$; $-(x+2)^4 \leq 0$ for any $x \in \Re$ and $-\sqrt{x} \leq 0$; $-\sqrt[4]{x} \leq 0$ for any $x \geq 0$. This implies that $-|x| + 4 \leq 4$; $3 - x^2 \leq 3$; $-\sqrt{x} - 1 \leq -1$ etc ..

  In addition to this sometimes you need to solve x in terms of y to find the range of a relation or function.

Next let us find the range of the following :

1. For $y = -3$: The range $= \{-3\}$.
2. For $x = 3$; $y = x - 2$; $y = x^3 + 1$ and $y = \sqrt[5]{x+4}$: The range $= (-\infty, \infty)$.
3. For $y = \sqrt{x+2} - 2$: The range $= [-2, \infty)$, because $\sqrt{x+2} \geq 0$.
4. For $y = 3 - \sqrt[4]{x+1}$: The range $= (-\infty, 3]$, because $-\sqrt[4]{x+1} \leq 0$.
5. For $y = |1 - 2x| - 4$: The range $= [-4, \infty)$, because $|1 - 2x| \geq 0$.
6. For $y = x^2 - 6x$: By completing to a perfect square $y = (x^2 - 6x + 9) - 9 = (x-3)^2 - 9 \geq -9$, so the range $= [-9, \infty)$.
7. For $y = \dfrac{1}{x-1}$: Notice that $x = \dfrac{1+y}{y}$, so $y \neq 0$ and then the range $= (-\infty, 0) \cup (0, \infty)$.
8. For $y = \sqrt{x^2+4}$: Notice that $\boxed{y \geq 0}$ and $x = \pm\sqrt{y^2-4}$, so $y^2 - 4 \geq 0 \Longrightarrow y^2 \geq 4$ $\Longrightarrow |y| \geq 2 \Longrightarrow y \leq -2$ or $y \geq 2$ but $y \geq 0$. Therefore the range $=$ $\boxed{(-\infty, -2] \cup [2, \infty)} \cap [0, \infty) = [2, \infty)$.
9. For $y = \sqrt{9 - x^2}$: Notice that $\boxed{y \geq 0}$ and $x = \pm\sqrt{9 - y^2}$, so $9 - y^2 \geq 0 \Longrightarrow y^2 \leq 9$ $\Longrightarrow |y| \leq 3 \Longrightarrow -3 \leq y \leq 3$ but $y \geq 0 \Longrightarrow$ the range $= [0, 3]$. We can also find the range of this function by noticing that the graph is the upper half of a circle of ceter $(0, 0)$, radius 3 .
10. For $y = -\sqrt{x^2 - 16}$: Notice that $\boxed{y \leq 0}$ and $x = \pm\sqrt{y^2 + 16}$. The only restriction on y is that $y \leq 0$, so the range $= (-\infty, 0]$.

Exercise :- Find the domain and range of the following :

1) $y = -|x - 8| + 8$

2) $y = \sqrt{x-2} - 3$

3) $x^2 - 4x + y^2 + 8y = 0$

4) $y = \sqrt{|x| - 4}$

5) $y = \dfrac{2x+1}{x-3}$

6) $y = \sqrt{\dfrac{x-5}{x+1}}$

7) $y = -\sqrt{x^2 - 16}$

8) $y = x^2 - x$

9) $y = \sqrt{x^2 - 6x + 9}$

10) $y = \sqrt{x^2 + 2x}$

11) $y = \dfrac{-1}{x^2 + 9}$

12) $y = \dfrac{\sqrt{x-5}}{\sqrt{x+1}}$

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