The absolute value $|x|$ of x : القيمة المطلقة للعدد س , absolute value bars $|\ |$: عَمُودَي القيمة المطلقة

The absolute value $|x|$ of a real number x is the **nonnegative** (zero or positive) value of x. This means that $|x|$ is either zero or positive, so $|x| \geq 0$.

Definition: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Again the definition says that when x is nonnegative, then $|x| = x$ and when x is negative, then $|x| = -x$ = -(negative) = positive. Therefore $|x| \geq 0$.

Example: $|5| = 5$, $|\pi| = \pi$, $|0| = 0$, $|-3| = -x = -(-3) = 3$, $|-7\sqrt{2}| = -x = -(-7\sqrt{2}) = 7\sqrt{2}$

Note The following statements are false (not always true): $|x| = x$, $-|x| = -x$, $|-x| = x$. This is because we don't know the **sign** (إشارة) of x. The simplest way to find the absolute value of any expression is by knowing the **sign** of that expression. If the expression is positive or zero, then its absolute value is **itself**. While, if it is negative, then its absolute value will be **− itself**.

Problem: Find the value of: $|\pi + 1| + |-\pi - 3| + |\pi - 4| + |\pi - 3| + |6 - 2\pi|$

Solution: Remember that $\pi \approx 3.14$. To evaluate, we need to know the sign of each expression inside the absolute value bars. Now $\pi + 1 > 0 \implies |\pi + 1| = \pi + 1$ and $-\pi - 3 < 0 \implies |-\pi - 3| = -(-\pi - 3) = \pi + 3$. Next $\pi < 4 \implies \pi - 4 < 0 \implies |\pi - 4| = -(\pi - 4) = 4 - \pi$ and $\pi > 3 \implies \pi - 3 > 0 \implies |\pi - 3| = \pi - 3$. Finally $3 < \pi \implies 6 < 2\pi \implies 6 - 2\pi < 0 \implies |6 - 2\pi| = -(6 - 2\pi) = 2\pi - 6$. Therefore the value $= (\pi + 1) + (\pi + 3) + (4 - \pi) + (\pi - 3) + (2\pi - 6) = 4\pi - 1$

Absolute value Laws: Let a, b, and c be real numbers

(1) $|a| \geq 0$ and $-|a| \leq 0$

(2) $|ab| = |a|\ |b|$

(3) $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$, $b \neq 0$

(4) $|a + b| \leq |a| + |b|$ (triangular inequality) ($|a + b| = |a| + |b|$ when a and b have the **same sign**)

(5) $|-a| = |a|$ (this is because $|-a| = |(-1)(a)| = |-1|\ |a| = |a|$)

(6) $|a - b| = |b - a|$ (because $|a - b| = |-(b - a)| = |b - a|$)

(7) $||a|| = |a|$ (the reason for this is $|a| \geq 0$)

Example: For $a \in \Re$, then $|a| + |-2a| - ||3a|| = |a| + |-2|\ |a| - |3a| = |a| + 2|a| - 3|a| = 3|a| - 3|a| = 0$

Example: If $x > 2$, then $\left|\dfrac{-3}{2 - x}\right| = \dfrac{|-3|}{|2 - x|} = \dfrac{3}{-(2 - x)} = \dfrac{3}{x - 2}$ (because $2 - x < 0$)

Example: $|3a - 4b| \leq |3a| + |-4b| = 3|a| + 4|b|$

Problem: If $1 < x < 2$, then find $\dfrac{|x - 1|}{x - 1} + \dfrac{|x - 2|}{x - 2}$

Solution: $x > 1 \implies x - 1 > 0 \implies |x - 1| = x - 1$ and $x < 2 \implies x - 2 < 0 \implies |x - 2| = -(x - 2)$.

Thus the value $= \dfrac{x - 1}{x - 1} + \dfrac{-(x - 2)}{x - 2} = 1 + (-1) = 0$

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Problem: True (T) or False (F):

Assume all letters used here are real numbers		
Statement	**T/F**	**Explanation**
$\|x\| = x$	F	$\|x\| = \|-2\| = x = -2$ is false
$\|a+3\| = a+3$	F	again $a+3$ is **not always positive**
$\|x^2+1\| = x^2+1$	T	here, x^2+1 is always positive
$\|-y-2\| = y+2$	F	also $y+2$ is **not always positive**
$\|\|b\|\| = b$	F	No, $\|\|b\|\| = \|b\|$ $(= b$ when $b \geq 0$ or $-b$ when $b < 0)$
if $x < 0$, then $\dfrac{x}{\|x\|} = -1$	T	$\dfrac{x}{\|x\|} = \dfrac{x}{-x} = -1$
$\|a+b\| = \|a\| + \|b\|$	F	for example, $3 = \|2+(-5)\| \neq \|2\| + \|-5\| = 7$
if $c \neq 0$, then $\dfrac{\|\|c\|\|}{\|-c\|} = 1$	T	$\dfrac{\|\|c\|\|}{\|-c\|} = \dfrac{\|c\|}{\|c\|} = 1$

Problem: If $x < -1$, then simplify the expression $|3x| + |-5x| + \Big||4x|\Big|$

Solution: $= 3|x| + 5|x| + 4|x| = 12|x| = 12(-x) = -12x$ ($|x| = -x$, because x is negative)

Problem: If $-4 < x < -2$, then write the expression $|x+4| + |2x+4| + \Big||-x|\Big|$ without absolute value bars.

Solution: $x > -4 \implies x+4 > 0 \implies |x+4| = x+4$, $x < -2 \implies x+2 < 0 \implies 2x+4 < 0 \implies$ $|2x+4| = -(2x+4)$ and $\Big||-x|\Big| = |x| = -x$. Thus the expression $= (x+4) - (2x+4) - x = -2x$

Problem: If $-1 < x < 1$, then write the expression $|x^2+1| + |x^2-1|$ without absolute value bars.

Solution: $x^2+1 > 0 \implies |x^2+1| = x^2+1$ and $x^2 < 1 \implies x^2-1 < 0 \implies |x^2-1| = -(x^2-1) =$ $1-x^2$. Therefore the expression $= (x^2+1) + (1-x^2) = 2$

Problem: If $0 < x < 1$, then write the expression $\left|\dfrac{2-2x}{|x|-|x-2|}\right|$ in simplest form.

Solution: Notice that $x > 0 \implies |x| = x$ and $x < 1 \implies 2x < 2 \implies 2-2x > 0 \implies |2-2x|$ $= 2-2x$. Also $x < 1 \implies x < 2 \implies x-2 < 0 \implies |x-2| = -(x-2) = 2-x$.

Therefore $\left|\dfrac{2-2x}{|x|-|x-2|}\right| = \dfrac{|2-2x|}{\big||x|-|x-2|\big|} = \dfrac{2-2x}{|x-(2-x)|} = \dfrac{2-2x}{|2x-2|} = \dfrac{2-2x}{-(2x-2)}$

$= \dfrac{2-2x}{2-2x} = 1$

Problem: If $-2 < x < 0$, then write the expression $\left|\dfrac{|x|+|x+2|}{x}\right|$ in simplest form.

Solution: Notice that $x < 0 \implies |x| = -x$ and $x > -2 \implies x+2 > 0 \implies |x+2| = x+2$.

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Thus $\left|\dfrac{|x|+|x+2|}{x}\right| = \dfrac{\big|\,|x|+|x+2|\,\big|}{|x|} = \dfrac{|-x+(x+2)|}{-x} = \dfrac{|2|}{-x} = -\dfrac{2}{x}$

Definition: If a and b are the **coordinates** of the points A and B on the real number line, then the **distance** between A and B is d(A,B) = d(a,b) = $|a-b|$ (or $= |b-a|$)

Example: If -5 , -2 , 3 are the **coordinates** of the points A , B , and C on the real number line, then find d(A,B) , d(B,C) , and d(A,C)

Solution: d(A,B) $= |a-b| = |-5-(-2)| = |-3| = 3$, d(B,C) $= |b-c| = |-2-3| = |-5| = 5$ and d(A,C) $= |a-c| = |-5-3| = |-8| = 8$.

Notice that d(A,B) + d(B,C) = d(A,C) , because the point B lies between the points A and C.

Problem: If $a<1$ and $a+2$, $-a+4$ are the **coordinates** of the points A , B on the real number line, then find d(A,B).

Solution: d(A,B) $= |(a+2)-(-a+4)| = |2a-2| = 2|a-1| = -2(a-1) = 2-2a$

Problem: Express each of the following statements mathematically:

Statement	Mathematical Expression
the distance between the numbers x and -3 is 5	$d(x,-3)=5 \Longrightarrow |x-(-3)| = |x+3| = 5$
the distance between a and 4 is **greater than** 1	$d(a,4)>1 \Longrightarrow |a-4|>1$
the number c is **at least** 3 units from the number 2	$d(c,2)\geq 3 \Longrightarrow |c-2|\geq 3$
the number b is **within** 4 units from -3	$d(b,-3)\leq 4 \Longrightarrow |b-(-3)| = |b+3|\leq 4$
x is **closer** to 0 than it is to -1	$d(x,0)<d(x,-1) \Longrightarrow |x-0|<|x-(-1)| \Longrightarrow |x|<|x+1|$
y is **farther** to 2 than it is to 6	$d(y,2)>d(y,6) \Longrightarrow |y-2|>|y-6|$
z is more than 1 unit from 2 but less than 8 units from 2	$1<d(z,2)<8 \Longrightarrow 1<|z-2|<8$

Important Note: Let n be a positive integer greater than 1 and x be any real number.

(i) If n is **even**, then $\sqrt[n]{x^n} = |x|$

(ii) If n is **odd**, then $\sqrt[n]{x^n} = x$

Problem: If $x<0$, then simplify $\sqrt{4x^2} + \sqrt[3]{8x^3} + \sqrt[4]{16x^4} + \sqrt[5]{-32x^5}$

Solution: $= 2|x| + 2x + 2|x| + (-2)(x) = -2x + 2x - 2x - 2x = -4x$

Problem: If $-1<x<1$, then simplify the expression $\sqrt{(x+1)^2} + \sqrt[4]{(x-1)^4}$

Solution: $= |x+1| + |x-1| = (x+1) - (x-1) = 2$ (because $x+1>0$ and $x-1<0$)

Problem: If $x<-3$, then simplify the expression $\sqrt{x^2+4x+4} + \sqrt[3]{x^3-3x^2+3x-1}$

Solution: $= \sqrt{(x+2)^2} + \sqrt[3]{(x-1)^3} = |x+2| + (x-1) = -(x+2) + (x-1) = -3$
(because $x+2<0$)