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Polynomial Equation in Radicals

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Abstract. Necessary and sufficient conditions for a radical class ρ of rings to satisfy the polynomial equation $\rho(R[x]) = (\rho(R))[x]$ have been investigated. The interrelationship of polynomial equation, Amitsur property and polynomial extensibility is given. It has been shown that complete analogy of R.E. Propes result for radicals of matrix rings is not possible for polynomial rings.

1. Introduction

Throughout this paper R will denote an associative ring and ω the universal class of all associative rings. We adopt the notation R_n and $R[x]$ to represent the ring of all $n \times n$ matrices and the ring of all polynomials over R in indeterminate x respectively. By a radical class ρ we mean a subclass of ω satisfying the following conditions:

(i) ρ is homomorphically closed.

(ii) Every ring R has a ρ-radical, i.e. $\forall R \in \omega$ there exists an ideal $\rho(R) \in \rho$ and for any ideal I of R, $I \in \rho \Rightarrow I \subseteq \rho(R)$.

(iii) The factor ring $R/\rho(R)$ is ρ-semisimple, i.e. $\rho(R/\rho(R)) = 0$.

A radical class ρ of rings is said to satisfy the matrix equation if $\rho(R_n) = (\rho(R))_n$, for every $R \in \omega$ where n is a natural number. By analogy we introduce the polynomial equation as

(PEq) $$\rho(R[x]) = (\rho(R))[x], \quad \forall R \in \omega.$$

R.E. Propes [10] has given necessary and sufficient conditions for a radical class

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to satisfy the matrix equation, whereas Aslam and Zaidi [2] have studied such conditions for upper radical, lower radical and sum of radical classes. The aim of this paper is to establish some necessary and sufficient conditions for the validity of (PEq) for a radical class. Closely related to our discussion is the Amitsur property. A radical class ρ is said to have this property if

(AP) $$\rho(R[x]) = (\rho(R[x]) \cap R)[x], \quad \forall R \in \omega.$$

Furthermore, we say that a radical class ρ is polynomially extensible if

(PE) $$R \in \rho \Rightarrow R[x] \in \rho.$$

In section 2 we investigate the interdependence of (PEq), (AP) and (PE). This leads us to our necessary conditions. Section 3 is devoted to the development of a sufficient condition for (PEq). In the process Anderson-Divinsky theorem has been generalized and a relationship between matrix and polynomial rings has been established.

By $I < R$, $I <_l R$ or $I <_r R$ it is meant that I is an ideal, a left ideal or a right ideal of R. The collection of all subrings of a ring R has been denoted by $\&(R)$.

For details of radical theory of rings we refer to Amitsur [1], Divinsky [5], Kurosh [8], Szasz [13] and Wiegandt [15].

2. Necessary conditions

We begin by analyzing the relationship of (AP) and (PE). We shall recall some characterizations of both these properties and give examples to show their mutual independence.

Proposition 2.1(Krempa [7]). *A radical class ρ has the Amitsur property if and only if $\rho(R[x]) \cap R = 0 \Rightarrow \rho(R[x]) = 0, \forall R \in \omega$.*

Proposition 2.2(Gardner [6]). *Let ρ be any radical class and let $\rho_x = \{R | R[x] \in \rho\}$. Then*

(i) *ρ_x is a radical class.*

(ii) *$\rho_x(R) \subseteq \rho(R[x]) \cap R, \forall R \in \omega$.*

(iii) *The radical class ρ is polynomially extensionable if and only if $\rho = \rho_x$.*

Examples 2.3.

(i) A ring R is said to be uniformly strongly prime, if there exists a finite subset F of R such that $aFb \neq 0$ for any nonzero elements $a, b \in R$. Here F is called a uniform insulator of R. The class of all uniformly strongly prime rings is known to be a special class. The upper radical u of this class is called the uniformly strongly prime radical (cf. [9]). Beidar, Puczylowski and Wiegandt [3] have shown that the class of all uniformly strongly prime rings is polynomially extensible and so is its upper radical u.

(ii) The Baer lower radical β, Levitzki radical L, Koethe (nil) radical N, Jacobson radical J and Brown-McCoy radical C have the Amitsur property (see [14]).

(iii) The radical classes β and L are polynomially extensible (cf. [14], Example 2.1(ii)).

(iv) Let R be a ring and R^0 be the corresponding ring with zero multiplication. For any radical class ρ the radical class ρ^0 defined by

$$\rho^0 = \left\{R | R^0 \in \rho\right\}$$

is polynomially extensible (see [14], Example 2.1 (ii)).

(v) Agata Smoktunowicz [12] constructed a nil ring R such that the polynomial ring $R[x]$ is not nil. Hence, the radical class N is not polynomially extensible.

(vi) Tumurbat and Wiegandt [14] have recently proved that the upper radical of all prime fields π is polynomially extensible but fails to satisfy the Amitsur property.

Corollary 2.4 *Polynomial extensibility and Amitsur property are independent.*

Proof. Follows from Example 2.3 (v) and (vi).

Now we shall study the interdependence of (i) (PEq) and (PE), and (ii) (PEq) and (AP). We shall draw the conclusion that (PEq) is a stronger property than both (PE) and (AP).

Theorem 2.5 *The following conditions are equivalent.*

(i) *ρ is polynomially extensible.*

(ii) $(\rho(R))[x] \subseteq \rho(R[x]), \forall R \in \omega$.

Proof. (i) $\Rightarrow$ (ii): Assume that (i) holds. Now $\rho(R) \in \rho \Rightarrow (\rho(R))[x] \in \rho$. But $(\rho(R))[x] < R[x]$, so by the maximality of $\rho(R[x])$ we have $(\rho(R))[x] \subseteq \rho(R[x])$.

(ii) $\Rightarrow$ (i): Let $R \in \rho$, then $\rho(R) = R \Rightarrow (\rho(R))[x] = R[x]$ and by (ii) $\rho(R[x]) = R[x]$ which shows that $R[x] \in \rho$. □

Corollary 2.6(Necessary Condition)**.** *If polynomial equation holds for a radical class ρ then ρ is polynomially extensible.*

Theorem 2.7. *A radical class ρ satisfies the polynomial equation if and only if it is polynomially extensible and has Amitsur property.*

Proof. (PEq) $\Rightarrow$ (AP): Suppose that ρ satisfies (PEq). Consider

$$\begin{aligned}(\rho(R[x]) \cap R)[x] &= ((\rho(R))[x] \cap R)[x] && \text{(by (PEq))}\\ &= (\rho(R))[x] = \rho(R[x]) && \text{(by (PEq))}\end{aligned}$$

(PEq) $\Rightarrow$ (AP): From Corollary 2.6.

(AP) and (PE) $\Rightarrow$ (PEq): Assume that ρ is polynomially extensible and satisfies Amitsur Property. Now Theorem 2.5 gives $(\rho(R))[x] \subseteq \rho(R[x])$ and the reverse inclusion follows from Amitsur Property. □

Example 2.8. In lieu of Corollary 2.6 and Theorem 2.7 a radical class ρ which either fails to satisfy (PE) or (AP) cannot satisfy (PEq). Thus the nil radical N and the radical π (Examples 2.3 (v) and (vi)) do not satisfy (PEq).

3. Sufficient condition

Let us first recall some well-known results of ring theory (see [4]).

Proposition 3.1. *Let S and T be subrings of a ring R, then*

(i) $(S \cap T)_n = S_n \cap T_n$

(ii) $(S \cap T)[x] = S[x] \cap T[x]$

Proposition 3.2. *Any ring R can be imbedded in both R_n and $R[x]$ as a subring.*

Proposition 3.3. *For any ring R with unity, $(R[x])[x] = R[x]$.*

Proof. The fact that $R[x] \subseteq (R[x])[x]$ is obvious. If R is a ring with unity then $x \in R[x]$ and so $(R[x])[x] \subseteq R[x]$.

The following isomorphism theorem is a pivotal result in the development of the theory of this paper.

Theorem 3.4. *For any ring R we have $R_n[x] \cong (R[x])_n$.*

Proof. Define $f : R_n[x] \to (R[x])_n$ by

$$f(\sum_{k=0}^{m}[a_{ij}^{(k)}]x^k) = [\sum_{k=0}^{m} a_{ij}^{(k)} x^k]$$

Evidently, f is a monomorphism. Also for arbitrary $\alpha = [\sum_{k=0}^{m_{ij}} a_{ij}^{(k)} x^k] \in (R[x])_n$, there exists $\beta = \sum_{k=0}^{m}[b_{ij}^{(k)}]x^k \in R_n[x]$ with $m = max\,\{m_{ij}\}_{i,j=1}^{n}$ and

$$b_{ij}^{(k)} = \begin{cases} a_{ij}^{k}, & if \;\; k \leq m_{ij} \\ 0, & otherwise \end{cases}$$

such that $f(\beta) = \alpha$. Thus f is surjective. □

A radical class ρ is said to be left hereditary (respectively right hereditary or hereditary) if $I <_l R$ (respectively $I <_r R$ or $I < R$) and $R \in \rho \Rightarrow I \in \rho$. Clearly, every left (right) hereditary radical is hereditary but not conversely. We say that a radical class ρ is left strong (respectively right strong) if $I <_l R$ (respectively $I <_r R$) and

$I \in \rho \Rightarrow I \subseteq \rho(R)$. A radical class ρ is strong if it is both left and right strong [10]. We introduce the terms super hereditary and super strong as follows.

Definition 3.5. A radical class ρ is said to be super hereditary if for any $S \in \&(R)$, $R \in \rho \Rightarrow S \in \rho$.

Definition 3.6. We say that a radical class ρ is super strong if $S \subseteq \rho(R)$, $\forall S \in \&(R) \cap \rho$.

This brings us to another important result.

Theorem 3.7(Generalized Anderson-Divinsky Theorem)**.** *A radical class ρ is super hereditary and super strong if and only if $\rho(S) = S \cap \rho(R)$, $\forall S \in \&(R)$.*

Proof. Since $S \cap \rho(R) \in \&(\rho(R))$ and ρ is super hereditary, therefore $S \cap \rho(R) \in \rho$. But $S \cap \rho(R) < S \Rightarrow S \cap \rho(R) \subseteq \rho(S)$ (by the maximality of $\rho(S)$). Also $\rho(S) \in \&(R) \cap \rho$ and ρ is super strong, so $\rho(S) \subseteq \rho(R) \Rightarrow \rho(S) \subseteq S \cap \rho(R)$. Hence, $\rho(S) = S \cap \rho(R)$.
Conversely, assume that $\rho(S) = S \cap \rho(R)$, $\forall S \in \&(R)$. If $R \in \rho$, then $\rho(R) = R$ and $\rho(S) = S \cap R = S \Rightarrow S \in \rho$. Thus ρ is super hereditary. Now if $S \in \&(R) \cap \rho$, then $S = \rho(S) = S \cap \rho(R) \subseteq \rho(R)$, showing that ρ is super strong. □

If R is a ring with unity then one can observe that $x \in R[x]$ and hence the following lemma can be concluded.

Lemma 3.8. *If R is a ring with unity then*

(i) $I < R \Rightarrow (I[x])[x] = I[x]$.

(ii) $M < R[x] \Rightarrow M[x] = M$.

We are now in a position to state our sufficient condition for rings with unity.

Theorem 3.9. *If ρ is a super hereditary and super strong radical class then $\rho(R[x]) = (\rho(R))[x]$, for any ring R with unity.*

Proof. Since $R[x]$ can be imbedded in $(R[x])_n \cong R_n[x]$ as a subring, therefore by Theorem 3.7

$$(*) \qquad \rho(R[x]) = R[x] \cap \rho(R_n[x])$$

Also R is imbeddable in $R_n[x]$, so

$$\begin{aligned} & \rho(R) = R \cap \rho(R_n[x]) \\ \Rightarrow \quad & \rho(R)[x] = (R \cap \rho(R_n[x]))[x] \\ & \qquad = R[x] \cap \rho(R_n[x])[x]. \qquad \text{(by Proposition 3.1(ii))} \end{aligned}$$

Since, $R_n[x]$ is a ring with unity and $(\rho(R_n))[x] < R_n[x]$, therefore $(\rho(R_n[x]))[x] = \rho(R_n[x])$ (by Lemma 3.8. ii).

Hence, $(\rho(R))[x] = R[x] \cap \rho(R_n[x]) = \rho(R[x])$ (by using $(*)$). □

This result can be extended to the class of all rings via Dorroh extension.

Theorem 3.10(Sufficient Condition for (PEq))**.** *If ρ is a super hereditary and super strong radical class then $\rho(R[x]) = (\rho(R))[x]$, $\forall R \in \omega$.*

Proof. Let R be a ring without unity and D be its Dorroh extension, then $R < D$ and so, by Theorem 3.7

$$\begin{aligned} \rho(R) &= R \cap \rho(D) \\ \Rightarrow \qquad (\rho(R))[x] &= (R \cap \rho(D))[x] \\ &= R[x] \cap (\rho(D))[x] \\ &= R[x] \cap \rho(D[x]) \qquad \text{(by Theorem 3.9)} \\ &= \rho(R[x]). \end{aligned}$$

□

It is worthwhile to compare this result with Propes condition for the validity of matrix equation.

Proposition 3.11(Propes [10])**.** *If ρ is a radical class of rings which is right or left hereditary and right or left strong, then $\rho(R_n) = (\rho(R))_n$, $\forall R \in \omega$.*

Remarks 3.12. The question may be asked whether or not it is possible to strengthen the hypothesis of Theorem 3.10 to the hypothesis of Proposition 3.11. The answer is: No. For instance consider Jacobson radical class J. It is well-known that J is left hereditary and left strong. But J fails to satisfy the polynomial equation. Thus the desired refinement is not possible. The same applies to right hereditary and right strong.

We now apply the theory developed previously to obtain an interesting result. First we recall the following proposition.

Proposition 3.13(Rowen [11, page 201])**.** *If $N(R) = 0$ then $J(R[x]) = 0$.*

Now a direct application of Theorem 3.4 yields

Theorem 3.14. *If $N(R) = 0$ then $J(R_n[x]) = 0$.*

Proof. By above proposition if $N(R) = 0$, then $J(R[x]) = 0$ which implies that $(J(R[x]))_n = 0$. Since J satisfies the matrix equation, therefore we have $J((R[x])_n) = 0$. Now by Theorem 3.4 $J(R_n[x]) = 0$.

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References

[1] S. A. Amitsur, *A general theory of radicals II, radicals in rings and bi-categories*, Amer. J. Math., **76**(1954), 100-125.

[2] M. Aslam and A. M. Zaidi, *Matrix equation in radicals*, Stud. Sci. Math. Hung., **28**(1993), 447-452.

[3] K. I. Beidar, E. R. Puczylowski and R. Wiegandt, *Radicals and polynomial rings*, J. Austral. Math. Soc., **72**(2002), 23-31.

[4] D. M. Burton, A first course in rings and ideals, Addison-Wesley, 1970.

[5] N. J. Divinsky, Rings and radicals, University of British Columbia, Vancouver, 1965.

[6] B. J. Gardner, *A note on radicals and polynomial rings*, Math. Scand., **31**(1972), 83-88.

[7] J. Krempa, *On the properties of polynomial rings*, Bull. Acad. Sci., **20**(1972), 545-548.

[8] A. G. Kurosh, *Radicals of rings and algebra*, (Russian) Math. Sbornik N.S., **33**(75), 13-26 (Translated in *Rings, Modules and radicals*, Proc. Colloq. Keozthely, 1297-1314, 1917) **MR 15 # 194g**.

[9] D. M. Olson, *A uniformly strongly prime radical*, J. Austral. Math. Soc., **43**(1987), 95-102.

[10] R. E. Propes, *The Radical equation* $\rho(R_n) = (\rho(R))_n$, Proc. Edin. Math. Soc., **14**(1975), 257-259.

[11] L. H. Rowen, Ring theory (Vol.1), Academic Press Inc., 1988.

[12] A. Smoktunowicz, *Polynomial rings over nil rings need not be nil*, J. Algebra, **233**(2000), 427-436.

[13] F. A. Szasz, Radical of rings, Mathematical Institute, Hungarian Academy of Sciences, 1981.

[14] S. Tumurbat and R. Wiegandt, *Radicals of polynomial rings*, Soochow J. Math., **29**(2003) 425-434.

[15] R. Wiegandt, Radicals and semisimple classes of rings, Queen papers in pure and Appl. Math. 37, Kingston, Ontario, 1974.