King Fahd Univesity of Petroleum & Minerals
Department of Mathematical Sciences

KEY

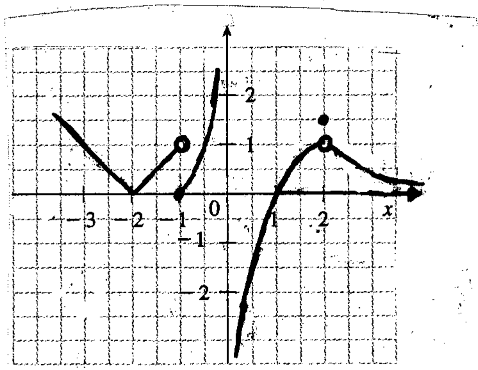
Math 101-Term 043
Exam I
Duration: 80 Minutes

Name: ______________________ Serial #: __________

ID #: ______________________ Section: __________

Question #	Mark	Total Mark
1		3
2		2
3		12
4		2
5		3
6		3
7		3
8		2
Total Marks		**30**

Q1. (3 pts) For the function graphed in the accompanying figure, find



a) $\lim_{x \to 2} f(x) = 1$ 0.5

b) $\lim_{x \to -1^-} f(x) = 1$ 0.5

c) $\lim_{x \to -1} f(x)$ Does not exist 0.5

$\lim_{x \to -1^+} f(x) = 0 \neq \lim_{x \to -1^-} f(x)$

d) $\lim_{x \to 0^+} f(x) = -\infty$ 0.5

e) $\lim_{x \to +\infty} f(x) = 0$ 0.5

f) x_0 that is a removable discontinuity. Explain your answer.

$x_0 = 2$ since $\lim_{x \to 2} f(x)$ exist $\neq f(2)$ 0.5

Q2. (2 pts) Use $\varepsilon - \delta$ definition to show that $\lim_{x \to -2} (2 - 3x) = 8$

Given $\epsilon > 0$, we want to find $\delta > 0$ such that

$|f(x) - 8| < \epsilon$ if $|x+2| < \delta$

$\Rightarrow |2 - 3x - 8| < \epsilon$ if $|x+2| < \delta$

$\Rightarrow |-3x - 6| < \epsilon$ if $|x+2| < \delta$

$\Rightarrow 3|x+2| < \epsilon$ if $|x+2| < \delta$

$\Rightarrow |x+2| < \epsilon/2$ if $|x+2| < \delta$

$\therefore \delta = \frac{\epsilon}{2}$

Q3. (12 pts) In problems (a) to (f) find the following limits if they exist, if not, when possible state whether the limit approaches $+\infty$ *or* $-\infty$

a) $\lim_{x\to 1} \frac{x^3+2x^2-x-2}{x^2+2x-3}$ $\quad \frac{0}{0}$

Sol:

$$\lim_{x\to 1} \frac{(x^3+2x^2)-(x+2)}{(x+3)(x-1)} = \lim_{x\to 1} \frac{x^2(x+2)-(x+2)}{(x+3)(x-1)}$$

$$= \lim_{x\to 1} \frac{(x+2)(x^2-1)}{(x+3)(x-1)}$$

$$= \lim_{x\to 1} \frac{(x+2)\cancel{(x-1)}(x+1)}{(x+3)\cancel{(x-1)}} = \frac{3(2)}{4} = \frac{3}{2}$$

OR

$$\begin{array}{r|rrrr} 1 & 1 & 2 & -1 & -2 \\ & & 1 & 3 & 2 \\ \hline & 1 & 3 & 2 & 0 \end{array}$$

$$\therefore \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+3x+2)}{\cancel{(x-1)}(x+3)} = \frac{3}{2}$$

b) $\lim_{x\to+\infty} \sqrt{\frac{3x^3-5x+7}{-x^2+x^3-x+1}}$

Sol.

$$\sqrt{\lim_{x\to+\infty} \frac{3\cancel{x^3}}{\cancel{x^3}}} = \sqrt{3}$$

c) $\lim_{x\to+\infty} (\sqrt{x^6+5}-x^3)$

Sol.

$$\lim_{x\to+\infty} (\sqrt{x^6+5}-x^3)\cdot\frac{\sqrt{x^6+5}+x^3}{\sqrt{x^6+5}+x^3} = \lim_{x\to+\infty} \frac{\cancel{x^6}+5-\cancel{x^6}}{\sqrt{x^6+5}+x^3}$$

$$= \lim_{x\to+\infty} \frac{5}{\sqrt{x^6(1+\frac{5}{x^6})}+x^3}$$

$$= \lim_{x\to+\infty} \frac{5}{|x^3|\sqrt{1+\frac{5}{x^6}}+x^3}$$

$$= \lim_{x\to+\infty} \frac{5}{x^3\sqrt{1+\frac{5}{x^6}}+x^3}$$

$$\frac{5}{\quad} = 0$$

d) $\lim_{x\to 2^+} \dfrac{\overset{+}{x}}{\underset{+}{x^2-4}}$

Sol. $+\infty$

e) $\lim_{x\to 0} \dfrac{x^2+x}{\sin 2x}$

Sol. $\lim_{x\to 0}\left[\dfrac{x^2}{\sin 2x}+\dfrac{x}{\sin 2x}\right] = \lim_{x\to 0}\left[\overset{0}{x}\cdot\cancel{\dfrac{x}{\sin 2x}} + \overset{1}{\cancel{\dfrac{2x}{\sin 2x}}}\cdot\dfrac{1}{2}\right] = \dfrac{1}{2}$

f) $\lim_{x\to 0} \dfrac{\tan x-\sin x}{\sin^3 x}$

Sol. $\lim_{x\to 0} \dfrac{\dfrac{\sin x}{\cos x}-\sin x}{\sin^3 x} = \lim_{x\to 0} \dfrac{\sin x-\sin x\cos x}{\cos x}\cdot\dfrac{1}{\sin^3 x}$

$= \lim_{x\to 0} \dfrac{\cancel{\sin x}(1-\cos x)}{\cos x}\cdot\dfrac{1}{\underset{\sin^2 x}{\cancel{\sin^3 x}}}$

$= \lim_{x\to 0} \dfrac{1-\cos x}{\cos x(1-\cos^2 x)}$

$= \lim_{x\to 0} \dfrac{\cancel{(1-\cos x)}}{\cos x\cancel{(1-\cos x)}(1+\cos x)} = \dfrac{1}{1(1+1)} = \dfrac{1}{2}$

Q4. (2pts) Show that the equation $5x^3+10x+8=0$ has at least one real solution in the interval $[-1,0]$.

Sol. Let $f(x) = 5x^3+10x+8$

* $f(x)$ is cont. everywhere (polynomial)
* $f(-1) = -5-10+8 < 0$

$f(0) = 8 > 0$

* By the "Intermediate Value Theorem" there is at least one real no. c. such that $f(c)=0$

$\Rightarrow 5c^3+10c+8=0$

Q5. (1+2 pts)Given $f(x)=\dfrac{x-2}{|x|-2}$

a) Find the values of x (if any) at which f is not continuous

$|x|-2=0 \Rightarrow |x|=2$ (1)

$\Rightarrow x=\mp 2$

b) Determine whether each such value is a removable discontinuity. Explain you answer.

at $x=-2$: $\lim\limits_{x\to -2} f(x) = \lim\limits_{x\to -2} \dfrac{x-2}{|x|-2} = \dfrac{-4}{0}$ (DNE) $\Rightarrow$ Not Removable (1)

at $x=2$: $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2} \dfrac{x-2}{|x|-2} = \lim\limits_{x\to 2} \dfrac{x-2}{x-2} = 1 \Rightarrow$ Removable (1)

Q6. (3 pts) Find a nonzero value for the constant k that makes $f(x)=\begin{cases} \dfrac{\tan kx}{x}, & x<0 \\ 3x+2k^2, & x\geq 0 \end{cases}$ continuous at $x=0$.

Sol.

f is cont. at $x=0 \Rightarrow \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x)$ ---- (1)

$\Rightarrow \lim\limits_{x\to 0^-} k\overset{1}{\dfrac{\tan kx}{kx}} = \lim\limits_{x\to 0}(3x+2k^2)$

$\Rightarrow k = 2k^2$

$\Rightarrow 2k^2-k=0$ (1)

$\Rightarrow k(2k-1)=0$

$\Rightarrow k=0$ or $k=\frac{1}{2}$

$\therefore$ the nonzero value of k is $\frac{1}{2}$. (1)

Q7. (3 pts) If $|f(x)+3| < 2|x-5|$, then find $\lim_{x\to 5} f(x)$. (Hint: Use the Squeezing theorem)

Sol.

$$\left.\begin{array}{ccccc} -2|x-5| & < & f(x)+3 & < & 2|x+5| \\ -3 & & -3 & & -3 \end{array}\right\} \quad (1)$$

$$-2|x-5|-3 < f(x) < 2|x+5|-3$$

as $x \to 5$: -3 and -3 $\quad$ (1)

By squeezing thm,

$$\therefore \lim_{x\to 5} f(x) = -3 \quad (1)$$

Q8.(2 pts) Find numbers a and b such that $\lim_{x\to 0} \frac{\sqrt{ax+b}-2}{x} = 1$

Sol.

* $\sqrt{b} - 2 = 0 \Rightarrow b = 4$ ---- (1)

* $$\lim_{x\to 0} \frac{\sqrt{ax+b}-2}{x} \cdot \frac{\sqrt{ax+b}+2}{\sqrt{ax+b}+2} = 1$$

$$\Rightarrow \lim_{x\to 0} \frac{ax+\cancel{4}-\cancel{4}}{x\left(\sqrt{ax+4}+2\right)} = 1 \qquad \boxed{\text{since } b=4}$$

$$\Rightarrow \lim_{x\to 0} \frac{a}{\sqrt{ax+4}+2} = 1$$

$$\Rightarrow \frac{a}{\sqrt{4}+2} = 1$$

$$\Rightarrow \frac{a}{4} = 1 \Rightarrow a = 4 \quad \text{----} (1)$$